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Aspects of noncommutative Lorentzian geometry for globally hyperbolic spacetimes.

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Abstract: Connes' functional formula of the Riemannian distance is generalized to the Lorentzian case using the so-called Lorentzian distance, the d'Alembert operator and the causal functions of a globally hyperbolic spacetime. As a step of the presented machinery, a proof of the almost-everywhere smoothness of the Lorentzian distance considered as a function of one of the two arguments is given. Afterwards, using a C^*-algebra approach, the spacetime causal structure and the Lorentzian distance are generalized into noncommutative structures giving rise to a Lorentzian version of part of Connes' noncommutative geometry. The generalized noncommutative spacetime consists of a direct set of Hilbert spaces and a related class of C^*-algebras of operators. In each algebra a convex cone made of self-adjoint elements is selected which generalizes the class of causal functions. The generalized events, called *loci*, are realized as the elements of the inductive limit of the spaces of the algebraic states on the C^*-algebras. A partial-ordering relation between pairs of loci generalizes the causal order relation in spacetime. A generalized Lorentz distance of loci is defined by means of a class of densely-defined operators which play the rôle of a Lorentzian metric. Specializing back the formalism to the usual globally hyperbolic spacetime, it is found that compactly-supported probability measures give rise to a non-pointwise extension of the concept of events.

1 Introduction.

1.1. *Some aspects of Connes' Riemannian non-commutative geometry.* Connes' noncommutative geometry is a very impressive coherent set of mathematical theories which encompass parts of mathematics born by very far and different contexts [2]. On the physical ground, applications of Connes' noncommutative geometry include general relativity, quantum field theory and many other research areas [2, 20, 11]. As regards the content of this paper we are interested in the approach of chapter VI of [2] (see also Chapter 6 of [20]). The basic ingredient introduced by

Connes to develop the analogue of differential calculus for noncommutative algebras is given by a so-called *spectral triple*, $(\mathcal{A}, \mathcal{H}, D)$. $\mathcal{A}$ is a unital algebra which is a subalgebra of the natural C^*-algebra of bounded operators on a Hilbert space $\mathcal{H}$. $D : \mathcal{D}(D) \to \mathcal{H}$ is a self-adjoint operator on $\mathcal{H}$, $\mathcal{D}(D) \subset \mathcal{H}$ being a dense linear manifold, such that the resolvent $(D - \lambda I)^{-1}$ is compact for each $\lambda \notin \mathbb{R}$. $[D, a]$ must be well defined at least as a quadratic form (see VI.1 in [2]) and bounded for every $a \in \mathcal{A}$.
Every smooth compact n-dimensional Riemannian manifold M equipped with a (Euclidean) spin structure determines a natural *commutative* (i.e., $\mathcal{A}$ is commutative) spectral triple. In that case $\mathcal{A}$ is the normed commutative unital involutive (the involution being the usual complex conjugation) algebra of Lipschitz[1] maps $f : M \to \mathbb{C}$, the norm being the usual sup-norm $||\cdot||_\infty$. $\mathcal{H}$ is the space $L^2(M, S)$ of the square integrable sections of the irreducible $\mathbb{C}^{2^{[\dim D/2]}}$-spinor bundle over M with measure $\mu_{\mathbf{g}}$ associated to the metric $\mathbf{g}$ on M. The positive Hermitean scalar product used to define L^2 reads

$$(\psi, \phi) := \int_M \psi^\dagger(x)\phi(x)\, d\mu_{\mathbf{g}}(x)\,.$$

This scalar product induces an operator norm which we denote by $||\cdot||_{\mathbf{L}(L^2(M,S))}$. Finally D is the Dirac operator associated with the Levi-Civita connection. It turns out that if $f \in \mathcal{A}$ is seen as a multiplicative operator, $||f||_\infty = ||f||_{\mathbf{L}(L^2(M,S))}$, $f^* = \overline{f}$, $1 = I$, where $1 : M \to \mathbb{C}$ is the constant map $1(x) = 1$. Therefore $\mathcal{A}$ is a subalgebra of the C^*-algebra of the bounded operators on $L^2(M, S)$ as it must be.
Remarkably, one can realize the topological and metric structure of the manifold in terms of the spectral triple only (see propositions 6.5.1 and 10.1.1 in [20]). Let us summarize this result. In the following $\overline{\mathcal{A}}$ denotes the (unital) C^*-algebra given by the completion of $\mathcal{A}$. M is homeomorphic to the space of (the classes of unitary equivalence of) irreducible representations of the C^*-algebra $\overline{\mathcal{A}}$, equipped with the topology of the pointwise convergence (also said Gel'fand's or $*$-weak topology). In the commutative case, the irreducible representations are unidimensional and coincide with the *pure algebraic states* on $\overline{\mathcal{A}}$. In this sense the points of M are pure algebraic states. All that is essentially due [2, 20, 11] to the well-known "commutative Gel'fand-Naimark theorem"[25]. In practice, $\overline{\mathcal{A}}$ turns out to be nothing but the C^*-algebra of the complex-valued continuous functions on M, $C(M)$ with the norm $||\cdot||_\infty$, and the pure state associated to any $p \in M$ trivially acts as $p(f) := f(p)$ for every $f \in C(M)$. As regards the metric, one has the functional formula

$$\mathbf{d}_E(x, y) = \sup\left\{|f(x) - f(y)| \;\mid\; f \in \mathcal{A},\; ||[D, f]|| \leq 1\right\}\ , \tag{1}$$

where $\mathbf{d}_E$ is the distance in the manifold which is induced by the metric. Notice that there is no reference to paths in the manifold, despite the left-hand side is defined as the infimum of the length of the paths from p to q:

$$\mathbf{d}_E(p, q) := \inf_{\Omega_{p,q}} \{L(\gamma)\}\,, \tag{2}$$

[1] I.e., for some $K_f \geq 0$, it holds $|f(p) - f(q)| \leq K_f \mathbf{d}_E(p, q)$ for every $p, q \in M$, $\mathbf{d}_E$ being the distance in M.

where $\Omega_{p,q}$ is the class of all continuous piecewise-smooth curves jointing p and q and $L(\gamma)$ is the Riemannian length of $\gamma \in \Omega_{p,q}$. As remarked by Connes [2], this fact is interesting on a pure physical ground. Indeed the path of quantum particles do not exist: wave functions exists but one must assume the existence of geometrical structures also discussing quantum particles. There is an analogous formula for the integration of functions $f \in \mathcal{A}$ over M based on the *Dixmier trace* tr_ω (below $c(n)$ is a coefficient depending on the dimension n of the manifold M only) [2, 20, 11],

$$\int_M f(x)\, d\mu_{\mathbf{g}}(x) = c(n)\ \mathrm{tr}_\omega \left(f|D|^{-n}\right) . \tag{3}$$

Whenever the algebra $\mathcal{A}$ of a spectral triple is taken noncommutative (1) can be re-interpreted as defining a distance in the space of pure states [2, 20, 11] and generalized interpretations are possible for (3). Similar noncommutative generalizations can be performed concerning much of differential and integral calculus finding out very interesting and useful mathematical structures giving rise to a remarkable interplay between mathematics and theoretical physics [2, 20, 11]. It is worth noticing that, for most applications, the Dirac operator D can be replaced by the Laplace-Beltrami one Δ as suggested in [7, 8] (see also [20]) and this is the way we follow within the present work.

Most physicists interested in quantum gravity believe that the Planck-scale geometry may reveal a structure very different from the geometry at macroscopic scales. This is a strong motivation for developing further any sort of noncommutative geometry. However, physics deals with *Lorentzian* spacetimes rather that *Euclidean*[2] spaces. To this end, the principal aim of this paper is the attempt to find the Lorentzian analogue of (1). Actually, we shall see that this is nothing but the first step in order to develop a noncommutative approach of the spacetimes causality.

1.2. *The Lorentzian puzzle.* The Lorentzian geometry, i.e. the geometry of spacetimes, is more complicated than the Euclidean one due to the presence of, local and global, *causal structures*. These take temporal and causal relations among events into account. The local, metrical and causal, structure is given by the Lorentzian metric. A physically relevant global causal structure is involved in the definition of a *globally hyperbolic* spacetime. Roughly speaking, a globally hyperbolic spacetime is a time-oriented Lorentzian manifold (that is a spacetime) which admits spacelike surfaces, called Cauchy surfaces, such that the assignment of Cauchy data on those surfaces determines the evolution of any field everywhere in the manifold if the field satisfies, for instance, Klein-Gordon equation. A globally hyperbolic spacetime seems to be the natural *scenario* where one represents the theory on the matter content of the universe, including (quantum) fields, elementary interactions and all that [28, 29]. In order to built up a Lorentzian noncommutative geometry, a generalization of the (local and global) *causal structure* of a spacetime is necessary. To make contact with Connes' program a natural question arises: What is the Lorentzian analogue of $\mathbf{d}_E$ to be used to generalize (1) in Lorentzian manifolds? An interesting object defined in either Euclidean and Lorentzian manifolds is the so-called *Synge world function* σ (see the Appendix A) which is related with the function $\mathbf{d}_E$ in Euclidean manifolds: Any

[2] We use "Euclidean" as synonym of "Riemanniann" throughout.

smooth, either Riemannian or Lorentzian, manifold is locally endowed with a smooth function $\sigma : N \times N \to \mathbb{R}$ where N is any convex normal neighborhood. σ maps $x, y \in N$ into one half the (signed) squared length of the unique geodesic segment, which joints x and y, contained in N. In Riemannian manifolds $\sigma \geq 0$. In Lorentzian manifolds, the sign is positive if and only if x, y are spatially separated, negative if and only if x, y are time-like related, and $\sigma(x, y) = 0$ for either $x = y$ or when x, y are null related. It is known that σ completely determines the metric at each point of the spacetime. In Euclidean manifolds $\mathbf{d}_E = \sqrt{2\sigma}$ holds whenever x, y belong to a common convex normal neighborhood, so, at least locally, it is possible to define $\mathbf{d}_E$ in terms of $\sqrt{2\sigma}$. However, any attempt to generalize (1) in Lorentzian manifolds by means of any analogue of $\mathbf{d}_E$ built up by means of σ faces the basic issue of the indefiniteness of the Lorentzian world function. $\mathbf{d} := \sqrt{2\sigma}$ would be complex-valued and so useless to restore some identity similar to (1). One could try to define $\mathbf{d}$ for spatially separated events only by taking the squared root of 2σ in that case. An immediate drawback is that the definition would not work whenever x and y are too far from each other since σ is not well defined outside convex normal neighborhoods. To avoiding the problem, one may try to use (2) for x, y spatially separated with $\Omega_{x,y}$ now denoting the class of *space-like* continuous piecewise smooth curves jointing x and y. This is not a nice idea too, because it would entail $\mathbf{d}(x, y) = 0$ (and thus also $\mathbf{d}(x, y) \neq \sqrt{2\sigma(x, y)}$) at least for x and y sufficiently close to each other and spatially separated. This is because, in convex normal neighborhoods, one may arbitrarily approximate null piece-wise smooth curves by means of piecewise smooth space-like curves with the same endpoints.

Actually several other problematic issues are related to the indefiniteness of $\mathbf{d}^2$. For instance, if D indicates the Dirac operator, the identity

$$||[D, f]|| = \operatorname{ess\,sup}_M |\mathbf{g}(df, df)| \,,$$

necessary to give rise to (1) (e.g., see [2, 20, 11]), fails to be fulfilled. This is because the left-hand side is not well-defined as a Hilbert-space operator norm since, in Minkowski spacetime (but this generalizes to any Lorentzian manifold equipped with a spin structure), the natural Lorentz invariant scalar product of spinors turns out to be indefinite. We do not address to these issues in the present work because we shall employ the Laplace-Beltrami-D'Alembert operator instead of the Dirac one (see [27] for another approach based on the Dirac operator and Krein spaces).

Another problematic technical issue related to the indefiniteness of the metric is the failure of the Lipschitz condition to define a valuable background algebra of functions $\mathcal{A}$. Indeed, in the Euclidean case $\mathbf{d}_E(p, \cdot)$ cannot be everywhere smooth but it turns out to be Lipschitz because of the triangular inequality (false in the Lorentzian case). The Lipschitz condition plays a relevant rôle in proving (1) and in the choice of the algebra $\mathcal{A}$ which contains $\mathbf{d}_E(p, \cdot)$. We remark that also the compactness of the manifold has to be dropped in the Lorentzian case because a compact spacetime contains a closed timelike curve (proposition 3.10 in [1]) and thus fails to be physical. The failure of the compactness gives rise to problems in the Euclidean case. However approaches to noncommutative Euclidean geometry exist in some cases [10]. If M is a Hausdorff locally compact space but it is not compact, there is a homeomorphism from M onto the space of complex homomorphism of the nonunital C^*-algebra of the complex functions on M which

vanish at infinity, $C_0(M)$, equipped with the pointwise-convergence topology [6]. So the points of M can be thought as multiplicative functionals on the C^*-algebra $\overline{\mathcal{A}} := C_0(M)$, and $\mathcal{A}$ can be taken as the algebra of complex continuous compactly-supported functions in M, $C_c(M)$. However, in the noncompact case (1) cannot be re-stated as it stands.

In the Lorentzian case, possible attempts to solve all these problems (also connected with Hamiltonian formulation of field theories including the gravitational field) [18, 19, 16, 15] are based on the foliation of the manifold by means of space-like hypersurfaces. On these hypersurfaces, provided they are compact (and endowed with spin structures), one can restore Connes' standard non-commutative approach referring to the Euclidean distance induced by the Lorentzian background metric. However, barring globally *static* spacetimes, any choice of the foliation is quite arbitrary. Moreover the relation between spatial spectral triples and causality seems to be quite involved. Finally, a classical background spacetime cannot completely eliminated through this way reducing possible attempts to formulate approaches to quantum gravity. Another approach to noncommutative Lorentzian geometry is presented in [27] in terms of Krein spaces. However the issue of the generalization of (1) is not investigated, but attention is focused on the generalization of (3) and the noncommutative differential calculus.

1.3. *A natural Lorentzian approach.* In this paper, first of all we show that there is a possible generalization of (1) in any physically well-behaved spacetime (see 1.5 for more details on the used definitions). In fact, in every *globally hyperbolic* spacetime M (i.e., a connected time oriented Lorentzian manifold which admits *Cauchy surfaces*) a functional identity similar to (1) arises which uses the so-called *Lorentzian distance* $\mathbf{d}(x,y)$ [1, 21], the class of almost-everywhere smooth *causal functions* and the Laplace-Beltrami-d'Alembert operator, locally $\Delta = \nabla^\mu\nabla_\mu$, associated to the Levi-Civita connection derivative ∇. (Actually the same result holds working with a vector fiber bundle $\mathfrak{F} \to M$ and more complicate second-order hyperbolic operators, see remark 1 after Theorem 3.1 below). The original idea to express the Lorentzian distance by a functional formula using the metric Laplacian was formulated by Parfinov and Zapatrin in [22] where part of the approach developed in the first part of this work was presented into a more elementary form without the requirement of global hyperbolicity.
Let us illustrate the ingredients pointed out above. Take $p, q \in M$. First suppose that $p \neq q$ and $p \preceq q$ which means that q belongs to the *causal future* of p (i.e., the subset of M of the events r such that there is a causal future-directed curve from p to r). In that case, the *Lorentzian distance* from p to q is defined as $\mathbf{d}(p,q) := \sup\{L(\gamma) \mid \gamma \in \Omega_{p,q}\}$, $\Omega_{p,q}$ denoting the set of all causal future-directed curves from p to q and $L(\gamma) \geq 0$ is the length of γ. $\mathbf{d}(p,q) := 0$ if either $p = q$ or $p \not\preceq q$. $\mathbf{d}$ enjoys an inverse triangular inequality if $p \preceq q \preceq r$: $\mathbf{d}(p,r) \geq \mathbf{d}(p,q) + \mathbf{d}(q,r)$. $\mathbf{d}$ is a natural object in *time oriented* Lorentzian manifolds, i.e., spacetimes, and it turns out to be continuous in *globally hyperbolic* spacetimes. $\mathbf{d}$ plays a crucial rôle in Lorentzian geometry [1, 21] because one can re-built the topology, the differential structure, the metric tensor and the time orientation of the spacetime by using $\mathbf{d}$ only, as we shall see shortly. If $N \subset M$ is open, a *causal functions* on N is a continuous map $f : N \to \mathbb{R}$ which does not decrease along every causal future-directed curve contained in N. $\mathcal{C}_{[\mu_\mathbf{g}]}(N)$ denotes the class of causal functions on $N \subset M$ which are smooth almost everywhere in N. $\mathcal{X}$ denotes the class of all regions I in the

spacetime M which are open, causally convex (i.e., if p, q belong to such a region, also every future-directed causal curve from p to q lies in the region) and such that $\overline{I}$ is compact, causally convex and ∂I has measure zero.
The Lorentzian equation which corresponds to (1) reads, in a globally hyperbolic spacetime M, for $p, q \in M$ with q in the causal future of p,

$$\mathbf{d}(p,q) = \inf \left\{ \langle f(q) - f(p) \rangle \;\middle|\; f \in \mathcal{C}_{[\mu_{\mathbf{g}}]}(\overline{I}),\ I \in \mathcal{X},\ p, q \in \overline{I},\ \left|\left| [f, [f, \not{\Delta}]]^{-1} \right|\right|_I \leq 1 \right\}, \tag{4}$$

where $\langle \alpha \rangle := \max\{0, \alpha\}$ for $\alpha \in \mathbb{R}$, $2\not{\Delta} := \Delta$, the latter being the Laplace-Beltrami-d'Alembert operator. $||\cdot||_S$ denotes the uniform norm of operators $A : L^2(S, \mu_{\mathbf{g}}) \to L^2(S, \mu_{\mathbf{g}})$ where $\mu_{\mathbf{g}}$ being the measure on $S \subset M$ naturally induced by the metric $\mathbf{g}$ of the spacetime. The restriction to a suitable class of compact sets $\overline{I}$ is useful to realize the events of the spacetime as pure states of unital C^*-algebras of functions containing the causal functions. It holds despite the manifold is not compact and these functions, in general, are not bounded on the whole manifold.
Afterwards we analyze, from the point of view of the C^*-algebras, the ingredients above showing that noncommutative generalizations are possible. In particular we introduce, in suitable algebraic context, the generalizations of the causal ordering relation $\preceq$ and of the Lorentzian distance. Specializing back to the commutative case, these generalizations give rise to a non-pointwise concept of event (compactly-supported probability measures on globally hyperbolic spacetimes) preserving the notion of causal ordering relation and Lorentzian distance.

1.4. *Structure of the work.* This paper is organized as follows. The remaining part of Section 1 contains basic definitions, notations and conventions used throughout the paper. In Section 2 we introduce the Lorentzian distance and the *causal functions* on a spacetime. More precisely (a) we present the basic properties of $\mathbf{d}$, (b) we show that it completely determines the structure of the spacetime and (c) we prove some propositions necessary to generalize (1). In particular we prove a theorem concerning the almost-everywhere smoothness of $\mathbf{d}$ in globally hyperbolic spacetimes. Section 3 is devoted to prove (4). Section 4 contains an algebraic analysis of the introduced mathematical structures and several generalizations. In particular (a) we introduce the concept of *locus* which generalizes the concept of event (or point in noncompact Euclidean manifolds) and we prove that (b) loci reduce to compactly supported (regular Borel) probability measures in the commutative case. Finally we show (c) that $\preceq$ and $\mathbf{d}$ can be extended into analogous mathematical objects related to the space of the loci which give rise to a noncommutative causality.

1.5. *Basic definitions, notations and conventions.* Throughout the work "iff" means "if and only if" and "smooth" means C^∞. Concerning differentiable manifolds we assume usual definitions. More precisely, a (n-dimensional) differentiable manifold M is a connected, Hausdorff, second countable topological space which is locally homeomorphic to $\mathbb{R}^n$ and is equipped with a C^∞-differentiable structure. Concerning differentiable functions in nonopen sets we give the following definition. If M is a differentiable manifold, $U \subset M$ is open and nonempty, and $V \subset \partial U$, $C^\infty(U \cup V)$ denotes the set of functions $f : U \cup V \to \mathbb{R}$ such that $f\restriction_U \in C^\infty(U)$ and, for every

$y \in V$, each derivative of any order, computed in a coordinate patch in some open neighborhood U_y of y, can be extended into a continuous function in $U_y \cap (U \cup V)$. We assume that the reader knows basic definitions and properties of manifolds equipped with (Lorentzian or Riemannian) metrics, Levi-Civita connection and geodesical flux. A.1 in Appendix A contains definitions and properties of the exponential map and the related mathematical machinery (convex normal neighborhoods).
We address the reader to [21, 1, 28, 23, 14] as general reference textbooks on spacetime structures. Let us summarize basic definitions, further definitions used in the paper will be given before relevant statements in the text. Appendix A contains a complete summary.
A (smooth) **Lorentzian manifold** $(M, \mathbf{g})$ is a $n \geq 2$-dimensional smooth manifold M with a smooth Lorentzian metric $\mathbf{g}$ (with signature $(-, +, \cdots, +)$). We use the following terminology concerning the classification of vectors and co-vectors. A vector $T \in T_x M$, $T \neq 0$, is said to be **space-like**, **time-like** or **null** if, respectively, $\mathbf{g}_x(T, T) > 0$, $\mathbf{g}_x(T, T) < 0$, $\mathbf{g}_x(T, T) = 0$. T is said to be **causal** if it is either time-like or null. The same terminology is used for co-vectors $\omega \in T^*_x M$ referring to $\uparrow\omega \in T_x M$, where $\mathbf{g}_x(\uparrow\omega, \cdot) = \omega$.
We remind the reader that a Lorentzian manifold $(M, \mathbf{g})$ is said to be **time orientable** if it admits a smooth non vanishing vector field $Z \in TM$ which is everywhere time-like. Afterwards a **time orientation**, $\mathcal{O}_t$, is the choice of one of the two equivalence classes of smooth time-like vector fields Z with respect to the equivalence relation $Z \sim Z'$ iff $\mathbf{g}(Z, Z') < 0$ everywhere. For each point $p \in M$, an orientation determines an analogous equivalence class of time-like vectors of $T_p M$, $\mathcal{O}_{tp}$. With the given definitions, a causal vector (co-vector) $T \in T_p M$ ($\omega \in T^*_p M$) is said to be **future directed** if $\mathbf{g}_p(Z(p), X) < 0$ ($\mathbf{g}_p(Z(p), \uparrow\omega) < 0$) and **past directed** if $\mathbf{g}_p(Z(p), X) > 0$ ($\mathbf{g}_p(Z(p), \uparrow\omega) > 0$).
A **spacetime** $(M, \mathbf{g}, \mathcal{O}_t)$ is a Lorentzian manifold $(M, \mathbf{g})$ which is time orientable and equipped with a time orientation $\mathcal{O}_t$; the points of M are also called **events**.
To conclude we give the definition of causal curves. In spacetime M, a piecewise C^1 curve (see A.5 for the detailed definition of **piecewise** C^k **curve** used in this work) γ is said to be **time-like**, **space-like**, **null**, **causal** if its tangent vector $\dot\gamma$ is respectively time-like, space-like, null, causal. Moreover, the curve is said to be **future** (**past**) **directed** if its tangent vector $\dot\gamma$ is **future** (**past**) **directed**.

2 Lorentzian distance and causal functions.

In this section we collect and review important notions and results in Lorentzian geometry, in particular focusing on the rôle of Lorentzian distance. Part of these results are well known but spread in the literature. A relevant result proven in Section 2.1 concerns the almost-everywhere smoothness of the Lorentzian distance (Theorem 2.1). In Section 2.2 is investigated the interplay of the Lorentzian distance and the notion of causal function in a spacetime. Finally, a preliminary formulation of the functional formula of the Lorentzian distance is presented (Theorem 2.2) using the built up machinery.

2.1. *The rôle and the properties of the Lorentzian distance in spacetimes.* To define the Lorentzian distance it is necessary to recall the notion of **Lorentzian length** of a (causal) curve. As is well known, the Lorentzian length $L(\gamma)$ of a piecewise C^1 curve $\gamma : [a, b] \to M$ is

$$L(\gamma) := \int_a^b \sqrt{|\mathbf{g}_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))|}\, dt\,. \tag{5}$$

Obviously the definition does not depend on the used parametrization.
It is convenient to extend the definition of Lorentzian length to *continuous causal curves* because several definitions and results of Lorentzian geometry found in the literature require the use of continuous causal curves. A continuous curve $\gamma : I \to M$ is said to be a **continuous future-directed causal curve** (see A.6) if the following requirement is fulfilled. For each $t \in I$, there is a neighborhood of t, I_t and a convex normal neighborhood of $\gamma(t)$, U_t, such that the following requirements are fulfilled. For $t' \in I_t \setminus \{t\}$, one has $\gamma(t') \neq \gamma(t)$ and (a) there is a future-directed causal (smooth) geodesic segment $\gamma' \subset U_t$ from $\gamma(t)$ to $\gamma(t')$ if $t' > t$ or (b) there is a future-directed causal (smooth) geodesic segment $\gamma' \subset U_t$ from $\gamma(t')$ to $\gamma(t)$ if $t' < t$. Similar definitions hold concerning **continuous future-directed timelike curves**, by replacing "causal" with "timelike" in the definitions above. The definition of $L(\gamma)$ is extended as follows ([14] p.214) to continuous future-directed causal curves γ. Suppose that γ, from p to q, is such that, for every open neighborhood U_γ of γ, there is a future-directed timelike piecewise C^1 curve γ' from p to q, then define $L_{U_\gamma}(\gamma) := \sup L(\gamma')$ varying γ' in U_γ as said. Then $L(\gamma) := \inf L_{U_\gamma}(\gamma)$ where U_γ varies in the class of all open neighborhoods of γ. If γ does not fulfill the initial requirement then γ must be an unbroken null geodesic (see [14] p.215) and thus one defines $L(\gamma) := 0$[3].

Remark. From now on a, either future-directed or past-directed, causal curve is supposed to be a continuous, respectively future-directed or past-directed, causal curve. Moreover continuous curves $\gamma : I \to M$ and $\gamma' : I' \to M$ are identified if there is an increasing homomorphism $h : I \to I'$ and $\gamma' \circ h = \gamma$.

Let us give the definition of Lorentzian distance. We remind the reader that, in a spacetime $(M, \mathbf{g}, \mathcal{O}_t)$, if $p, q \in M$, $p \preceq q$ means that either $p = q$ or there is a future-directed causal curve from p to q, whereas $p \prec q$ means that $p \preceq q$ and $p \neq q$, and finally $p \prec\!\!\prec q$ means that there is a future-directed time-like curve from p to q. ($\prec\!\!\prec$ and $\preceq$ are clearly transitive relations moreover, if $p, q, r \in M$, $p \prec\!\!\prec q$ and $q \preceq r$ entail $p \prec\!\!\prec r$, similarly $p \preceq q$ and $q \prec\!\!\prec r$ entail $p \prec\!\!\prec r$ [23].)

Definition 2.1. *Let $(M, \mathbf{g}, \mathcal{O}_t)$ be a spacetime. If $p, q \in M$ and $\Omega_{p,q}$ denotes the class of the future-directed causal curves from p to q, the* **Lorentzian distance from** p **to** q, $\mathbf{d}(p, q) \in$

[3] A maybe equivalent definition can be given noticing that a continuous future directed causal curve γ satisfies a local Lipschitz condition (with respect to the coordinates of a sufficiently small neighborhood of each point of γ) and thus it is almost everywhere differentiable. So, one defines $L(\gamma)$ using (5) too (see [1] p. 136).

$[0, +\infty) \cup \{+\infty\}$ *is [21, 1]*

$$\mathbf{d}(p,q) := \begin{cases} \sup\{L(\gamma) \mid \gamma \in \Omega_{p,q}\} & \text{if } p \prec q, \\ 0 & \text{if } p \not\prec q. \end{cases} \tag{6}$$

Remarks (1) By the given definition of $L(\gamma)$, $\mathbf{d}(p,q) = \sup\{L(\gamma)\}$ attains the same value if one restricts the range of γ to the piecewise C^1 curves of $\Omega_{p,q}$.
(2) Differently from the Euclidean case, in general $\Omega_{p,q} \neq \Omega_{q,p}$, and thus $\mathbf{d}(p,q) \neq \mathbf{d}(q,p)$.

The Lorenz distance enjoys several relevant properties which will be useful later. Proposition 2.1 below presents the elementary properties of the Lorentzian distance in relation with the *causal sets* of a spacetime. From now on we use the following definitions of causal sets in a spacetime $(M, \mathbf{g}, \mathcal{O}_t)$. The topological and causal properties of these sets which are employed in the work are presented in A.9, A.11, A.12 and A.23. If $S \subset M$,

$J^+(S) :=\{q \in M \mid p \preceq q$ for some $p \in S\}$ is the **causal future** of S ,
$J^+(S) :=\{q \in M \mid q \preceq p$ for some $p \in S\}$ is the **causal past** of S ,
$I^+(S) :=\{q \in M \mid p \prec\!\!\prec q$ for some $p \in S\}$ is the **chronological future** of S ,
$I^-(S) :=\{q \in M \mid q \prec\!\!\prec p$ for some $p \in S\}$ is the **chronological past** of S .

Moreover $I(p,q) := I^+(p) \cap I^-(q)$ and $J(p,q) := J^+(p) \cap J^-(q)$. $p, q \in M$ are said to be **time related**, if either $I^+(p) \cap I^-(q) \neq \emptyset$ or $I^-(p) \cap I^+(q) \neq \emptyset$, **causally related** if either $J^+(p) \cap J^-(q) \neq \emptyset$ or $J^-(p) \cap J^+(q) \neq \emptyset$. Causally related events $p, q \in M$, $p \neq q$, which are not time related are called **null related**. $S, S' \subset M$ are said to be **spatially separated** if $(J^+(S) \cup J^-(S)) \cap S' = \emptyset$ (which is equivalent to $(J^+(S') \cup J^-(S')) \cap S = \emptyset$).
We remind the reader that a set S of a spacetime M is **causally convex** when $J(p,q) \subset S$ if $p, q \in S$ (see A.11 and for properties of causally convex sets and strongly causal spacetimes). A spacetime is **strongly causal** when every event admits a fundamental set of open neighborhoods consisting of causally convex sets. A spacetime is called **chronological** if there are no events p, q such that $p \prec\!\!\prec q \prec\!\!\prec p$ (equivalently, it does not contain any closed future-directed timelike curve). Finally, a **globally hyperbolic** spacetime (see also A.16-A.23 and the end of 8.3 in [28] about possible equivalent definitions) is a strongly-causal spacetime $(M, \mathbf{g}, \mathcal{O}_t)$ such that every $J(p,q)$ is either empty or compact for each pair $p, q \in M$ (see A.12-A.15 for further definitions and properties, here we remind the reader only that a globally hyperbolic spacetime is both strongly causal and chronological).

Proposition 2.1. *If $(M, \mathbf{g}, \mathcal{O}_t)$ is a spacetime and $p, q, r \in M$:*
(a) $I^+(p) = \{q \in M \mid \mathbf{d}(p,q) > 0\}$. *Moreover, if $p \neq q$ and both $\mathbf{d}(p,q)$ and $\mathbf{d}(q,p)$ are finite, then either $\mathbf{d}(p,q) = 0$ or $\mathbf{d}(q,p) = 0$;*
(b) *if $p \preceq q \preceq r$, the* **inverse triangular inequality** *holds, that is*

$$\mathbf{d}(p,r) \geq \mathbf{d}(p,q) + \mathbf{d}(q,r) \; ; \tag{7}$$

(c) $\mathbf{d}$ *is lower semicontinuous on* $M \times M$*;*
(d) *if* U_p *is a, sufficiently small, convex normal neighborhood of* p*,* $\mathbf{d}(p,\cdot)\restriction_{U_p \cap J^+(p)}$ *is finite, belongs to the class* $C^\infty(U_p \cap J^+(p))$ *and, for all* $q \in U_p \cap J^+(p)$*,*

$$\sigma(p,q) = -\frac{1}{2}\mathbf{d}(p,q)^2\,; \tag{8}$$

where $\sigma(p,q)$ *is one half the squared geodesic distance from* p *to* q*, also called Synge's world function, defined by using the exponential map (see A.0 in Appendix A).*
(e) *if* $p \preceq q$ *and there is a curve* $\gamma \in \Omega_{p,q}$ *with* $L(\gamma) = \mathbf{d}(p,q)$ *(i.e.,* γ *is* **maximal***), then* γ *can be re-parametrized to be a smooth geodesic.*

If $(M, \mathbf{g}, \mathcal{O}_t)$ *is globally hyperbolic it also holds:*
(f) $J^+(p) = \overline{\{q \in M \mid \mathbf{d}(p,q) > 0\}}$*;*
(g) $\mathbf{d}$ *is finite;*
(h) $\mathbf{d}$ *is continuous on* $M \times M$*;*
(i) *if* $p \prec q$*, there is a causal geodesic from* p *to* q*,* γ *with* $L(\gamma) = \mathbf{d}(p,q)$*.*

Proof. Items **(a)**,**(c)**,**(e)**,**(g)**,**(h)** are proven in Section 4.1 of [1]. **(b)** is a trivial consequence of the definition of $\mathbf{d}$. Concerning **(d)**, everything is a consequence of the smoothness of σ and of (8). The latter can be proven noticing that the length from p of causal geodesic segments through p, in a convex normal neighborhood is maximal (proposition 4.5.3 in Section 4.5 of [14]) and using theorem 4.27 in [1]. **(f)** is a consequence of (a) and A.12. The proof of **(i)** can be found in [21] p.411. $\Box$

A very remarkable result of Lorentzian geometry is that the Lorentzian distance determines the whole, local and global (topological, differential, metric), structure of a spacetime as summarized in Proposition 2.2.

Proposition 2.2. *Let* $(M, \mathbf{g}, \mathcal{O}_t)$ *be a spacetime with Lorentzian distance* $\mathbf{d}$ *and* $n := \dim M$*.*
(a) *If* M *is strongly causal (in particular if* M *is globally hyperbolic), its topology is generated by the sets* $\{x \in M \mid \mathbf{d}(p,x)\cdot\mathbf{d}(x,q) > 0\}$ *for all pairs* $p, q \in M$ *with* $p \ll q$ *(we assume that* $0\cdot\infty = \infty\cdot 0 = 0$*).*
(b) *There is an atlas of* M*,* $\{(U_p, \varphi_p)\}_{p\in M}$*,* U_p *being an open neighborhood of* p *with coordinate maps given by* $\varphi_p : q \mapsto (\mathbf{d}(p_1,q), \ldots, \mathbf{d}(p_n,q)) \in \mathbb{R}^n$*,* $p_1, p_2, \ldots, p_n$ *being suitable events about* p*.*
(c) *For every pair of smooth vector fields* X, Y *and every event* $p \in M$ *it holds*

$$\mathbf{g}_p(X_p, Y_p) = -\frac{1}{2}\lim_{p \ll q \to p} X_q(Y_q(\mathbf{d}(p,q)^2))\,. \tag{9}$$

(d) *If* M *is chronological (in particular if* M *is globally hyperbolic),* $T_p \in T_pM$ *is timelike future-directed iff* $\mathbf{d}(p, \exp_p(tT_p)) > 0$*,* $t \in (0,u]$ *for some* $u > 0$*.*
(e) *Let* $(M', \mathbf{g}', \mathcal{O}'_t)$ *be another spacetime with Lorentzian distance* $\mathbf{d}'$*. If* M *is strongly causal (in particular if* M *is globally hyperbolic) and* $f : M \to M'$ *(not necessarily continuous) is surjective and* $\mathbf{d}'(f(p), f(q)) = \mathbf{d}(p,q)$ *for all* $p, q \in M$*, then* f *is a diffeomorphism (and thus* a fortiori *a*

homeomorphism), preserves the metric, i.e. $f^*\mathbf{g}' = \mathbf{g}$, and preserves the time orientation.

Proof. **(a)** See the end of A.11. **(b)** Let $n := \dim M$. Fix $p \in M$ and a sufficiently small convex normal neighborhood U of p. Take a basis of T_p^*M made of future directed co-vectors ω_k, $k = 1, \ldots, n$, considers n geodesics γ_k through p, with respectively tangent vectors $\uparrow\omega_k$ and take n events $p_k \in \gamma_k \cap U \cap I^-(p)$. The maps $x \mapsto \mathbf{d}(p_k, x)$ are smooth in a neighborhood of p by (d) of Proposition 2.1. Using that proposition and (29) one gets $d\mathbf{d}(p_k, x)|_p = \beta_k \omega_k$ (there is no summation over k) for some reals $\beta_k \neq 0$. Since the co-vectors ω_k are linearly independent, such a requirement is preserved by the vectors $d\mathbf{d}(p_k, x)$ in a neighborhood of p and the maps $x \mapsto \mathbf{d}(p_k, x)$ define an admissible coordinate map about p. **(c)** In a Riemannian normal coordinate system centered on p, $\sigma(p, q) = (1/2) g_{ab}(p) x_q^a x_q^b$. Hence $\mathbf{g}_p(X_p, Y_p) = \lim_{q \to p} X_q(Y_q(\sigma(p, q)))$ by direct computation. The limit does not depend on the used curve because $q \mapsto X_q(Y_q(\sigma(p, q)))$ is continuous about p. Using γ from p to some $q_0 \in I^+(p)$ with $\gamma \setminus \{p\} \subset I^+(p)$, (d) of Proposition 2.1 implies (9). **(d)** If T_p is time-like and future-directed, $t \mapsto \exp_p(tT)$ is a timelike future directed curve, thus $\exp_p(tT) \in I^+(p)$ if $t > 0$ and the thesis is a consequence of (a) of Proposition 2.1. Conversely, if $T_p \in T_pM$ and $\mathbf{d}(p, \exp_p(tT_p)) > 0$ when $t \in (0, u]$ for some $u > 0$ then $exp_p(tT_p) \in I^+(p)$ in that interval for (a) of Proposition 2.1. Taking $t_0 < u$, $t_0 > 0$ sufficiently small, there is a convex normal neighborhood U_p containing either p, $q := \exp_p(t_0 T_p)$ and $\exp_p(tT_p)$ for $t \in (0, t_0]$. Theorem 8.1.2 in [28] implies that the unique geodesic in U_p from p to q must be timelike and thus T_p is such. If T_p were past directed, $t \mapsto \exp_p(tT_p)$ would be such giving $I^+(p) \cap I^-(p) \neq \emptyset$ which violates the chronological condition. **(e)** One has to prove the injectivity of f only, because the proof of the remaining items is a direct consequence of (a)-(d). The preservation of the Lorentz distance implies that $p \lll q$ in M iff $f(p) \lll f(q)$ in M'. Then suppose $p \neq q$ in M and $f(p) = f(q)$. Let V be an open causally convex neighborhood of p with $q \notin V$. Take $q_1, q_2 \in V$ with $q_1 \lll p \lll q_2$. It holds $I^+(q_1) \cap I^-(q_2) \subset V$ and thus $q \notin I^+(q_1) \cap I^-(q_2)$. However $f(q_1) \lll f(p) = f(q) \lll f(q_2)$ implies $q_1 \lll q \lll q_2$ and $q \in I^+(q_1) \cap I^-(q_2)$ which is a contradiction. $\Box$

Remark. The item (e) can be made stronger (see theorem 4.17 in [1]) proving that if $(M', \mathbf{g}', \mathcal{O}'_t)$ is another spacetime with Lorentzian distance $\mathbf{d}'$, $(M, \mathbf{g}, \mathcal{O}_t)$ is strongly causal and $f : M \to M'$ (not assumed to be continuous) is surjective and for some constant $\beta > 0$, $\mathbf{d}'(f(p), f(q)) = \beta \mathbf{d}(p, q)$ for all $p, q \in M$, then f is a diffeomorphism and satisfies $f^*\mathbf{g}' = \beta \mathbf{g}$.

We can state the first important technical result of this section in Theorem 2.1. The theorem concerns some features of the structure of the *cut locus* in Lorentzian geometry and establishes that the Lorentzian distance is almost-everywhere smooth if considered as a function of one of the two arguments. These properties, in turn, will be used to prove the functional formula of the Lorentzian distance (in particular they are useful to prove Proposition 2.3).
To understand the statement of the theorem we remind the reader that a subset X of a manifold M is said to **have measure zero** if for every local chart (U, ϕ), the set $\phi(U \cap X) \subset \mathbb{R}^{\dim(M)}$ has Lebesgue measure zero. When M is endowed with a nondegenerate smooth metric $\mathbf{g}$, it turns out that $X \subset M$ has measure zero if and only of it has measure zero with respect to the

positive complete Borel measure $\mu_{\mathbf{g}}$ induced by $\mathbf{g}$ on M.
Some further preliminary definitions and results concerning the *nonspace-like cut locus* are necessary. We use notations and definitions in chapter 9 of [1]. Consider $p \in M$, with M globally hyperbolic. Let $\mathbf{h}$ be a complete Riemannian metric[4] on M. Define

$$UM := \{v \in TM \,|\, \mathbf{h}(v,v) = 1,\ \mathbf{g}(v,v) \leq 0,\ v \text{ is future directed}\}, \quad UM_p := \{v \in UM \,|\, \pi(v) = p\}.$$

If $v \in UM$, $t \mapsto c_v(t)$, with $t \in [0,a)$, denotes the unique geodesic starting from $p = \pi(v)$ with initial tangent vector given by v and maximal domain. Finally define, for $v \in UM$,

$$s_1(v) := \sup\{t > 0 \,|\ \ c_v(t) \text{ is maximal form } p \text{ to } c_v(t)\}\,.$$

"$c_v(t)$ is maximal from p to $c_v(t)$" means [1] that $L(c_v\restriction_{[0,t]}) = \mathbf{d}(p, c_v(t))$. Using (b) of Proposition 2.1, it arises that if a future-directed causal geodesic segment $\gamma : [a,b] \to M$ is maximal, then $\gamma\restriction_{[a',b']}$ is so for $a \leq a' < b' \leq b$. Notice that $s_1(v) > 0$ in strongly causal spacetimes and thus in globally hyperbolic spacetimes because, in these spacetimes, every geodesic is maximal in a convex normal neighborhood containing the initial point [1]. It is known (see proposition 9.33 in [1]) that s_1 is lower semicontinuous in globally hyperbolic spacetimes and, if (1) the spacetime is globally hyperbolic, (2) $s_1(v)$ is finite and (3) c_v extends to $[0, s_1(v)]$, then s_1 is continuous in v. Finally define

$$\Gamma^+_{ns}(p) := \{s_1(v)v \,|\, v \in UM_p, s_1(v) < +\infty,\ c_v \text{ extends to } [0, s_1(v)]\} \text{ and } C^+(p) := \exp(\Gamma^+_{ns}(p))\,.$$

The second definition is consistent because c_v extends to $[0, s_1(v)]$ iff it is defined in some maximal domain $[0, s_1(v) + \epsilon)$, $\epsilon > 0$ and this is equivalent to saying that $c_{s_1(v)s}$ is defined in some maximal domain $[0, 1 + \frac{\epsilon}{s_1(v)})$. Therefore if $v \in UM_p$, "$s_1(v) < +\infty$ and c_v extends to $[0, s_1(v)]$" is equivalent to "$s_1(v)v \in U_p$" and so

$$\Gamma^+_{ns}(p) = \{s_1(v)v \mid v \in UM_p, s_1(v)v \in U_p\}\,.$$

$C^+(p)$ is a subset of $J^+(p)$ by construction and it is called the **future nonspace-like cut locus of** p. If $s_1(v)v \in \Gamma^+_{ns}(p)$, $\exp(s_1(v)v)$ is called the **future cut point of** p along c_v. The **past nonspace-like cut locus** is defined similarly, with the obvious changes. Everything can be re-stated for the past nonspace-like cut locus with the necessary obvious replacements. By theorem 9.35 of [1], in globally hyperbolic spacetimes, $C^+(p)$ is closed (and thus $J^+(p) \setminus C^+(p)$ is the union of the open set $I^+(p) \setminus C^+(p)$ and $\partial I^+(p) \setminus C^+(p) \subset \partial\left(I^+(p) \setminus C^+(p)\right))$.

Theorem 2.1. *Let $(M, \mathbf{g}, \mathcal{O}_t)$ be a globally hyperbolic spacetime and take any $p \in M$. The following statements hold.*
(a) $\partial I^+(p) = \partial J^+(p) = J^+(p) \setminus I^+(p)$ *and* $C^+(p) \subset J^+(p)$ *are closed, without internal points, with measure zero;*

[4] It exists on any differentiable Hausdorff second-countable manifold as proven by Nomizu and Ozeki, *Proc. Amer. Math. Soc.* **12**, 889-891.

(b) $J^+(p) \setminus (C^+(p) \cup \partial J^+(p)) = I^+(p) \setminus C^+(p)$ *is open and homeomorphic to* $\mathbb{R}^{dim(M)}$*;*
(c) $\exp_p$ *defines a diffeomorphism onto* $I^+(p) \setminus C^+(p)$ *with domain given by an open subset of* T_pM *of the form*

$$\mathcal{A}_p = \{X \in T_pM \mid X \textit{is timelike and future directed}, 0 < |\mathbf{g}_p(X,X)| < \lambda_X \textit{for some } \lambda_X > 0\}\,;$$

(d) $\mathbf{d}(p,\cdot)^2$ *belongs to* $C^\infty(J^+(p) \setminus C^+(p))$ *and* $\mathbf{d}(p,\cdot)$ *belongs to* $C^\infty(I^+(p) \setminus C^+(p))$*;*
(e) $\mathbf{d}(p,\cdot)$ *satisfies the* **timelike eikonal equation** *for* $q \in I^+(p) \setminus C^+(p)$*,*

$$\mathbf{g}_q(\uparrow d_q \mathbf{d}(p,q), \uparrow d_q \mathbf{d}(p,q)) = -1.$$

Proof. See the Appendix C. □

Remarks. **(1)** The statement and the proof of the item (b) are known in the literature [1].
(2) $C^\infty(J^+(p) \setminus C^+(p))$ is valid in the sense of 1.5. Indeed since $C^+(p)$ is closed, $I^+(p)$ is open and $J^+(p) = \overline{I^+(p)}$ (see A.12), one has that $J^+(p) \setminus C^+(p) = (I^+(p) \setminus C^+(p)) \cup (\partial I^+(p) \setminus C^+(p))$ where $I^+(p) \setminus C^+(p)$ is open and $\partial I^+(p) \setminus C^+(p) \subset \partial(I^+(p) \setminus C^+(p))$.
(3) Due to the possibility of reversing the time orientation preserving the globally hyperbolicity, it turns out that, fixing the latter argument of $\mathbf{d}(p,q)$ and varying the former, one gets a function in $C^\infty(J^-(q) \setminus C^-(q))$ and the analogues of items (a)-(e) above hold. Finally $q \in C^+(p)$ iff $p \in C^-(q)$ as a consequences of theorems 9.12 and 9.15 in [1].

2.2. *Causal functions and Lorentzian distance.* We introduce a lemma and a proposition necessary to generalize (1) to Lorentzian manifolds in terms of the Lorentzian distance. To this end, we have to give some introductory definitions in particular concerning so-called *causal functions*. The introduced machinery, together with the results achieved in Section 2.1 will make us able to present a preliminary version of the formula of the Lorentzian distance in globally hyperbolic spacetimes (Theorem 2.2).

Definition 2.2. *Let* $(M, \mathbf{g}, \mathcal{O}_t)$ *be a spacetime. Let* $N \subset M$ *such that* $N = A \cup B$ *where* A *is open and* $B \subset \partial A$*. A continuous function* $f : N \to \mathbb{C}$ *is said to be* **essentially smooth on** N *if there is a closed set* $C_f \subset N$ *with measure zero, such that* $f\restriction_{N \setminus C_f}$ *is smooth.* $\mathcal{E}_{[\mu_\mathbf{g}]}(N)$ *indicates the class of such functions.*

Definition 2.3. *Let* $(M, \mathbf{g}, \mathcal{O}_t)$ *be a spacetime. Let* $N \subset M$*. A continuous function* $f : N \to \mathbb{R}$ *is either a* **causal function** *or a* **time function** *on* N *if, respectively, it is non-decreasing or increasing along every future-directed causal curve contained in* N*.* $\mathcal{C}(N)$ *and* $\mathcal{T}(N)$ *respectively denote the class of causal functions and the class of time functions on* N*. If* N *is taken as in Def. 2.2,* $\mathcal{C}_{[\mu_\mathbf{g}]}(N) := \mathcal{E}_{[\mu_\mathbf{g}]}(N) \cap \mathcal{C}(N)$*,* $\mathcal{T}_{[\mu_\mathbf{g}]}(N) := \mathcal{E}_{[\mu_\mathbf{g}]}(N) \cap \mathcal{T}(N)$*.*

Remark. Notice that $\mathcal{T}(N) \subset \mathcal{C}(N)$. Moreover, if $N \subset M$ is taken as in Def. 2.2 and M is globally hyperbolic, $\mathcal{T}(N) \cap C^\infty(N) \neq \emptyset$ because a smooth time function exists on the whole manifold M (see A.13 and A.15). In general spacetimes $\mathcal{C}(N) \cap C^\infty(N) \neq \emptyset$ because the constant

functions are causal functions.

The following technical lemma and a proposition are useful in generalizing (1). The proposition states that, in suitable domains, $\mathbf{d}$ defines a natural causal/time function which is also essentially smooth.

Lemma 2.1. *In a globally hyperbolic spacetime $(M, \mathbf{g}, \mathcal{O}_t)$ with Lorentzian distance $\mathbf{d}$, take an open causally convex (A.11) set $I \subset M$ such that ∂I has measure zero. If $f \in \mathcal{C}_{[\mu_\mathbf{g}]}(\overline{I})$ then, df is either 0 or causal and past directed in an open set $J \subset \overline{I}$ with $\mu_{(\mathbf{g})}(J) = \mu_\mathbf{g}(I) (= \mu_\mathbf{g}(\overline{I}))$ and*

$$\operatorname{ess\,inf}\{|d_z f| \mid z \in \overline{I}\} \leq \inf\left\{ \frac{f(y) - f(x)}{\mathbf{d}(x,y)} \;\middle|\; x, y \in \overline{I},\, x \ll y \right\} . \tag{10}$$

Above, $\leq$ can be replaced by $=$ if $f \in \mathcal{T}_{[\mu_\mathbf{g}]}(\overline{I})$.

Proof. See the Appendix C. $\square$

Proposition 2.3. *Let $(M, \mathbf{g}, \mathcal{O}_t)$ be a globally hyperbolic spacetime let $\mathbf{d}$ indicate the corresponding Lorentzian distance and, for each $p \in M$, define the functions $f_p(\cdot) := \mathbf{d}(p, \cdot)$ and $h_p(\cdot) := -\mathbf{d}(\cdot, p)$. It holds*
(a) *$f_p, h_p \in \mathcal{E}_{[\mu_\mathbf{g}]}(M)$;*
(b) *$f_p\restriction_{I^+(p)} \in \mathcal{T}_{[\mu_\mathbf{g}]}(I^+(p))$ and $h_p\restriction_{I^-(p)} \in \mathcal{T}_{[\mu_\mathbf{g}]}(I^-(p))$;*
(c) *$f_p\restriction_N \in \mathcal{C}_{[\mu_\mathbf{g}]}(N)$ and $h_p\restriction_N \in \mathcal{C}_{[\mu_\mathbf{g}]}(N)$ for every $N \subset M$ as in Def. 2.2.*

Proof. We prove the thesis for f_p, the other case is analogous. **(a)** is a direct consequence of Theorem 2.1. and the fact that $f_p(x) = 0$ if $x \notin J(p)$. **(b)** Let $\gamma \subset I^+(p)$ be a causal future-directed curve. Take $x, y \in \gamma$ with $x = \gamma(t), y = \gamma(t')$ and $t' > t$. We want to show that it holds $f_p(x) < f_p(y)$, i.e., $\mathbf{d}(p, y) \leq \mathbf{d}(p, x)$ is not possible. Notice that $y \neq x$ because the spacetime is globally hyperbolic and thus causal, in fact we have $p \ll x \prec y$ (and thus $p \ll y$). Suppose that $\mathbf{d}(p, x) \geq \mathbf{d}(p, y)$. By (b) of Proposition 2.1 it must also hold $\mathbf{d}(p, y) \geq \mathbf{d}(p, x) + \mathbf{d}(x, y)$. Putting together and using $\mathbf{d}(x, y) \geq 0$ one gets

$$0 \leq \mathbf{d}(x, y) \leq \mathbf{d}(p, y) - \mathbf{d}(p, x) \leq 0 \, .$$

The only chance is $\mathbf{d}(x, y) = 0$ and $\mathbf{d}(p, y) = \mathbf{d}(p, x)$. Since the spacetime is globally hyperbolic, there must be a future-directed maximal null geodesic γ_2 from x to y by (i) of Proposition 2.1. By the same item there must be a time-like maximal future-directed geodesic γ_1 from p to x. $\gamma_1 * \gamma_2$ is a causal future-directed curve from p to y. Moreover it holds $L(\gamma_1 * \gamma_2) = \mathbf{d}(p, x) + 0 = \mathbf{d}(p, y)$. By (e) in Proposition 2.1, $\gamma_1 * \gamma_2$ can be re-parametrized into a maximal geodesic from p to y which must be time-like, since $\mathbf{d}(p, y) > 0$, y being in $I^+(p)$. This is impossible since γ_2 is null. **(c)** If $N \cap J^+(p) = \emptyset$ the proof is trivial since f_p is constant on N. Suppose that $N \cap J^+(p) \neq \emptyset$ and that $\gamma \subset N$ is a future-directed causal curve with $\gamma(u) \in J^+(p)$ for some u, the remaining cases being trivial. In these hypotheses $\gamma(u') \in J^+(p)$ for $u' > u$ because of A.7. Then there are

various cases to be analyzed for $t < t'$ where we use the fact that f_p vanishes outside $I^+(p)$ by Proposition 2.1. (1) if $\gamma(t), \gamma(t') \not\in J^+(p)$, the thesis holds because $0 = f_p(\gamma(t)) \leq f_p(\gamma(t')) = 0$. (2) If $\gamma(t) \not\in J^+(p)$ and $\gamma(t') \in J^+(p)$ the thesis holds because $0 = f_p(\gamma(t)) \leq f_p(\gamma(t')) \geq 0$. (3) If $\gamma(t), \gamma(t') \in I^+(p)$, the thesis holds by (a). (4) $\gamma(t), \gamma(t') \in \partial I^+(p) = \partial J(p)$. In that case $f_p(\gamma(t)) = f_p(\gamma(t')) = 0$ by (a) and (f) of Proposition 2.1. (4) $\gamma(t) \in \partial I^+(p)$ and $\gamma(t') \in I^+(p)$, in that case $0 = \gamma(t) < \gamma(t')$ by (a) and (f) of Proposition 2.1. The case $\gamma(t') \in \partial I^+(p)$ and $\gamma(t) \in I^+(p)$ is forbidden because $p \lll \gamma(t) \preceq \gamma(t')$ implies $p \lll \gamma(t')$ by the remark in A.7. $\Box$

The last technical proposition necessary to state the preliminary version of the functional formula of the Lorentzian distance concerns the interplay of relatively-compact causally-convex sets in globally hyperbolic spacetimes and essentially smooth causal functions.
In A.16-A.23 of Appendix A the definition of *Cauchy surface* and the relevant properties of these surfaces are given. An important results of Lorentzian geometry (see A.20) states that: *a spacetime $(M, \mathbf{g}, \mathcal{O}_t)$ is globally hyperbolic iff it admits a Cauchy surface.* This statement can be adopted as an equivalent definition of a globally hyperbolic spacetime (see remark in the end of 8.3 in [28] for a proof of equivalence of the various definitions of globally hyperbolicity). In a globally hyperbolic spacetime M, if $S \subset M$ is a smooth Cauchy surface and $p \in J^+(S)$, $I(S,p)$ and $J(S,p)$ respectively denote $I^-(p) \cap I^+(S)$ and $J^-(p) \cap J^+(S)$. One can straightforwardly prove that $I(s,p)$ is not empty iff $p \in I^+(p)$. It is not very difficult to show that $I(S,p)$ and $J(S,p)$ are causally convex. A.8 implies that $I(S,p)$ is open and $I(S,p) \subset J(S,p)$. The sets $I(p,S)$ and $J(p,S)$ enjoy analogous properties.

Proposition 2.4. *Let $(M, \mathbf{g}, \mathcal{O}_t)$ be a globally hyperbolic spacetime and let $\mathcal{X}$ denote the class of open, nonempty, causally-convex subsets of M, I, such that $\overline{I}$ is compact, causally convex and ∂I has measure zero. The following statements hold.*
(a) *The class $\mathcal{X}$ is a* **covering** *of M, i.e., $\bigcup \mathcal{X} = M$, and defines a* **direct set** *with respect to the set-inclusion partial-ordering relations, i.e., if $A, B \in \mathcal{X}$ there is $C \in \mathcal{X}$ such that $A \cup B \subset C$.*
(b) *If $A \in \mathcal{X}$, $\mathcal{T}_{[\mu_\mathbf{g}]}(A) \neq \emptyset$ and $\mathcal{C}_{[\mu_\mathbf{g}]}(\overline{A}) \neq \emptyset$.*
(c) *If $p, q \in M$, $p \preceq q$ iff $f(p) \leq f(q)$ for all $f \in \mathcal{C}_{[\mu_\mathbf{g}]}(\overline{I})$ and $I \in \mathcal{X}$ such that $p, q \in \overline{I}$.*
(d) *(i) If $S \subset M$ is a smooth Cauchy surface for M and either $p \in I^+(S)$ or $p \in I^-(S)$, respectively $I(S,p) \in \mathcal{X}$ and $\overline{I(S,p)} = J(S,p)$ or $I(p,S) \in \mathcal{X}$ and $\overline{I(p,S)} = J(p,S)$.*
(ii) If $p \in M$, there is a fundamental system of neighborhoods of p made of sets $I(r,s) \in \mathcal{X}$ with $\overline{I(r,s)} = J(r,s)$.

Proof. See the Appendix B. $\Box$.

Now we are able to state and prove the second important theorem of this section which is nothing but a preliminary version of the functional formula of the Lorentz distance.
From now on we use the following notation: If $T \in T_pM$, $|T| := \sqrt{|\mathbf{g}_p(T,T)|}$, similarly, if $\omega \in T^*_pM$, $|\omega| := \sqrt{|\mathbf{g}_p(\uparrow\omega, \uparrow\omega)|}$.

Theorem 2.2. *Let $(M, \mathbf{g}, \mathcal{O}_t)$ be a globally hyperbolic spacetime and $p, q \in M$. Defining $\langle\alpha\rangle := \max\{0, \alpha\}$ for all $\alpha \in \mathbb{R}$, it holds*

$$\mathbf{d}(p,q) = \inf\{\langle f(q) - f(p)\rangle \mid f \in \mathcal{C}_{[\mu_\mathbf{g}]}(\overline{I}),\ I \in \mathcal{X},\ p, q \in \overline{I},\ \operatorname{ess\,inf}_{\overline{I}}|df| \geq 1\}\,. \tag{11}$$

Proof. Define $\mu(p,q) := \inf\{\langle f(q) - f(p)\rangle \mid f \in \mathcal{C}_{[\mu_\mathbf{g}]}(\overline{I}),\ I \in \mathcal{X},\ p, q \in \overline{I},\ \operatorname{ess\,inf}_{\overline{I}}|df| \geq 1\}$.
We want to show that $\mu(p,q) = \mathbf{d}(p,q)$. First consider the case $p \preceq q$.
To this end consider the map $f_p : x \mapsto \mathbf{d}(p,x)$, where $x \in \overline{I_{f_p}}$ with $I_{f_p} = I(p,S)$, S being a smooth Cauchy surface with $p, q \in I^-(S)$. Theorem 2.1 and Proposition 2.3 say that such a f_p can be used to evaluate $\mu(p,q)$ because all of the necessary requirements are fulfilled. We trivially have $0 \leq \mathbf{d}(p,q) = f_p(q) - f_p(p) = \langle f_p(q) - f_p(p)\rangle$ and thus $\mu(p,q) \leq \mathbf{d}(p,q)$. To conclude, it is sufficient to show that $\mu(p,q) \geq \mathbf{d}(p,q)$. By Lemma 2.1, if $I \in \mathcal{X}$, $f \in \mathcal{C}_{[\mu_\mathbf{g}]}(\overline{I})$ and $\operatorname{ess\,inf}_{\overline{I}}|df| \geq 1$, we have

$$\inf\left\{ \frac{f(y) - f(x)}{\mathbf{d}(x,y)} \;\middle|\; x, y \in \overline{I},\ x \prec\!\!\prec y \right\} \geq 1\,.$$

Therefore, in $\overline{I}$, $x \prec\!\!\prec y$ entails $\langle f(y) - f(x)\rangle \geq \mathbf{d}(x,y)$. The inequality holds also if $x \preceq y$ because, by (a) and (f) of Proposition 2.1, if $x \preceq y$ and $x \prec\!\!\prec y$ is false, it must be $\mathbf{d}(p,q) = 0$. In that case $\langle f(y) - f(x)\rangle \geq \mathbf{d}(x,y)$ is trivially true. In particular, if $p, q \in \overline{I}$ and $p \preceq q$, then $0 \leq \mathbf{d}(p,q) \leq \langle f(q) - f(p)\rangle$. By the definition of μ, this implies $\mu(p,q) \geq \mathbf{d}(p,q)$.
Let us consider the case $q \preceq p$. Similarly to above, take $f_q : x \mapsto \mathbf{d}(q,x)$ in some $J(q,S)$ with $p, q \in I^-(S)$. f_q can be used to compute $\mu(p,q)$ obtaining $f_q(q) - f_q(p) \leq 0$ which implies $\langle f_q(q) - f_q(p)\rangle = 0$ and thus $\mu(p,q) = 0$ because $0 \leq \mu(p,q) \leq \langle f_q(q) - f_q(p)\rangle$ by definition.
Finally consider the case of p and q spatially separated. In that case it is possible to find (see below) two, sufficiently small, regions $I(x,y), I(x',y')$ with $p \in I(x,y)$, $q \in I(x',y')$ and such that $\overline{I(x,y)} = J(x,y)$ and $\overline{I(x',y')} = J(x',y')$ are spatially separated. We conclude that $A := I(p,y) \cap I(q,y') \in \mathcal{X}$. Then $x \mapsto f(x) := \mathbf{d}(p,x) + \mathbf{d}(q,x)$ defines an element of $\mathcal{C}_{[\mu_\mathbf{g}]}(\overline{A})$ and satisfies $\mathbf{g}(df, df) = -1$ a.e. by construction, hence it can be used to evaluate $\mu(p,q)$ producing $\mu(p,q) = 0 = \mathbf{d}(p,q)$ because $\langle f(q) - f(p)\rangle = \langle 0 - 0\rangle = 0$ and $0 \leq \mu(p,q) \leq \langle f(q) - f(p)\rangle$.
Let us prove the existence of $I(x,y), I(x',y')$ with the properties above. Since $\{q\} \cap (J^+(p) \cup J^-(p)) = \emptyset$ and $J^+(p) \cup J^-(p)$ is closed (A.12), there is a neighborhood of q, V which satisfies $V \cap (J^+(p) \cup J^-(p)) = \emptyset$. As the spacetime is strongly causal, V can be fixed with the form $I(x',y')$. By a suitable restriction (A.8) it is possible to fix $J(x',y')$ such that $q \in I(x',y')$ and $J(x',y') \cap (J^+(p) \cup J^-(p)) = \emptyset$. This is equivalent to $\{p\} \cap (J^+(J(x',y')) \cup J^-(J(x',y'))) = \emptyset$. A.12 implies that $J^+(J(x',y')) \cup J^-(J(x',y'))$ is closed because, since the spacetime is globally hyperbolic, $J(x',y')$ is compact. Using the same way followed above, one can find $I(x,y)$ such that $p \in I(x,y)$ and $J(x,y) \cap (J^+(J(x',y')) \cup J^-(J(x',y'))) = \emptyset$. We have proven that there are two regions $I(x,y), I(x',y')$ with $p \in I(x,y)$, $q \in I(x',y')$ and $J(x,y)$, $J(x',y')$ are spatially separated. $\square$

3 The functional formula of the Lorentzian distance.

3.1. *Laplace-Beltrami-d'Alembert operator and the net of Hilbert spaces.* The results achieved in Section 2 allow us to generalize (1) in a globally hyperbolic spacetime $(M, \mathbf{g}, \mathcal{O}_t)$ using the Lorentzian distance $\mathbf{d}$. The procedure consists of a translation of the statement of Theorem 2.2, Eq.(11) in particular, in terms of operators. To this end, a preliminary discussion on the remaining ingredients (operators) which appear in (4) is necessary.
Consider the class of Hilbert spaces $L^2(\overline{I}, \mu_{\mathbf{g}})$, $I \in \mathcal{X}$. These spaces are naturally isomorphic to closed subspaces of $L^2(M, \mu_{\mathbf{g}})$. $||\cdot||_{\mathbf{L}(L^2(\overline{I}))}$ denotes the uniform norm operator in the corresponding $L^2(\overline{I}, \mu_{\mathbf{g}})$. In those spaces three classes of useful operators can be defined: the operators Δ_I which are obtained by means of a suitable restriction of the Laplace-Beltrami-d'Alembert operator, the functions $f \in \mathcal{C}_{[\mu_{\mathbf{g}}]}(\overline{I})$ viewed as multiplicative operators and the commutators $[f, [h, \Delta_I]]$.

Definition 3.1. *Let $(M, \mathbf{g}, \mathcal{O}_t)$ be a globally hyperbolic spacetime. Referring to the notations above, the* **Laplace-Beltrami-d'Alembert operator** *on $L^2(M, \mu_{\mathbf{g}})$, is*

$$\Delta : C_0^\infty(M) \to L^2(M, \mu_{\mathbf{g}}) ,$$

with $\Delta := \nabla_\mu \nabla^\mu$ in local coordinates, ∇ denoting the Levi-Civita covariant derivative. Δ_I denotes the restriction of Δ to the, dense in $L^2(\overline{I}, \mu_{\mathbf{g}})$, linear manifold $C^\infty(\overline{I})$, $I \in \mathcal{X}$.

As a general remark we notice that Δ is densely defined, symmetric and admit self-adjoint extensions because it commutes with the complex conjugation, conversely every Δ_I is not symmetric because it is not Hermitean in the considered domain because of nonvanishing boundary terms.
Let us pass to consider the causal functions and commutators. Every $f \in \mathcal{C}_{[\mu_{\mathbf{g}}]}(\overline{I})$ $(I \in \mathcal{X})$ can be seen as a multiplicative (self-adjoint) operator in $L^2(\overline{I}, \mu_{\mathbf{g}})$ with domain given by the whole space $L^2(\overline{I}, \mu_{\mathbf{g}})$. The commutator $[f, [h, \Delta_I]]$ is well-defined as an operators in $L^2(\overline{I}, \mu_{\mathbf{g}})$ with the domain and the properties stated below. A remarkable step which permits to translate (11) into (4) is the identity established by Eq. (12) in the item (b) below.
In the following, if A is an operator in a Hilbert space with scalar product $(\cdot, \cdot)$, $A \leq \alpha I$ means $\alpha I - A \geq 0$, i.e., $(\Psi, (\alpha I - A)\Psi) \geq 0$ for all Ψ in the domain of A.

Lemma 3.1. *In a globally hyperbolic spacetime $(M, \mathcal{O}_t, \mathbf{g})$ take $I \in \mathcal{X}$ and $f, h \in \mathcal{C}_{[\mu_{\mathbf{g}}]}(\overline{I})$. Let $\mathcal{D}_{I,f,g} := C_0^\infty(I \setminus (S_f \cup S_h))$, S_t being the set of singular points of $t \in T_{[\mu_{\mathbf{g}}]}(I)$.*
(a) *$\mathcal{D}_{I,f,g} \subset L^2(\overline{I}, \mu_{\mathbf{g}})$ is a dense linear manifold, invariant for either f, h, Δ_I.*
(b) *Δ_I and $[f, [h, \Delta_I]]$ are symmetric on $\mathcal{D}_{I,f,g}$, the latter operator is also essentially self-adjoint on $\mathcal{D}_{I,f,g}$ and*

$$[f, [h, \Delta_I]] = 2\mathbf{g}(\uparrow df, \uparrow dh) \quad \text{almost everywhere in } \overline{I} \,. \tag{12}$$

(c) *The following equivalent relations hold*
(i) $\sigma(\overline{[f, [h, \Delta_I]]}) \subset (-\infty, 0]$,

(ii) $[f,[h,\Delta_I]] \leq 0$ *on* $\mathcal{D}_{I,f,g}$,
(iii) $\overline{[f,[h,\Delta_I]]} \leq 0$.

Proof. **(a)** It is obvious that $\mathcal{D}_{I,f,g}$ is a linear manifold in $L^2(\overline{I},\mu_{\mathbf{g}})$. It is also dense therein because, as $S_f \cup S_h$ is closed, $I\setminus(S_f\cup S_h)$ is open and $C_0^\infty(I\setminus(S_f\cup S_h))$ is dense in $L^2(I\setminus(S_f\cup S_h),\mu_{\mathbf{g}})$ which coincides with $L^2(\overline{I},\mu_{\mathbf{g}})$ because $S_f\cup S_h\cup\partial I$ has measure zero. The invariance properties can be proven by direct inspection. **(b)** Δ_I restricted to the linear manifold $\mathcal{D}_{I,f,g}$ is Hermitean by construction (notice that $I\setminus(S_f\cup S_h)$ is open) and thus it is symmetric too because the domain is dense. (12) can be proven by direct inspection on $I\setminus(S_f\cup S_h)$. $[f,[h,\Delta_I]] = 2\mathbf{g}(\uparrow df,\uparrow dh)$ entails the Hermiticity (and thus the symmetry, the domain being dense) because $\mathbf{g}(\uparrow df,\uparrow dh)$ is a real measurable function which acts as a multiplicative operator. However the symmetry also follows form standard properties of the commutator and the symmetry of the operators f,h,Δ_I. The essentially self-adjointness of $[f,[h,\Delta_I]]$ on $\mathcal{D}_{I,f,g}$ is assured by Nelson's theorem [25] proving that $\mathcal{D}_{I,f,g}$ is made by analytic vectors. The proof is immediate using the fact that $\mathbf{g}(\uparrow df,\uparrow dh)$ is smooth and thus bounded when restricted to any compact set contained in $I\setminus(S_f\cup S_h)$. **(c)** By Lemma 2.1, df and df are almost everywhere causal and past directed where they do not vanish, therefore $[f,[h,\Delta_I]] = \mathbf{g}(\uparrow df,\uparrow dh) \leq 0$ almost everywhere. In turn it entails (ii), namely $(\Psi,[f,[h,\Delta_I]]\Psi) \leq 0$ for all $\Psi\in\mathcal{D}_{I,f,g}$. Let us prove the equivalence of (i),(ii) and (iii). The unique self-adjoint extension of $[f,[h,\Delta_I]]$ coincides with the closure of the same operator and thus (ii) implies (iii). Moreover (iii) implies (ii) trivially. Using the spectral measure of $\overline{[f,[h,\Delta_I]]}$ one trivially see that (i) is equivalent to (iii). $\Box$

3.2. *The functional formula of the Lorentz distance.* To conclude, we can state and prove the formula (4) which generalizes (1) in globally hyperbolic spacetimes.

Theorem 3.1. *Let* $(M,\mathbf{g},\mathcal{O}_t)$ *be a globally hyperbolic spacetime with Lorentzian distance* $\mathbf{d}$ *and define* $\not\Delta_I := \frac{1}{2}\Delta_I$ *and* $\langle\alpha\rangle := \max\{0,\alpha\}$ *if* $\alpha\in\mathbb{R}$. *The Lorentzian distance of* $p,q\in M$ *can be computed as follows*

$$\mathbf{d}(p,q) = \inf\left\{\langle f(q)-f(p)\rangle \,\middle|\, f\in\mathcal{C}_{[\mu_{\mathbf{g}}]}(\overline{I}),\ I\in\mathcal{X},\ p,q\in\overline{I},\ \left|\left|\overline{[f,[f,\not\Delta_I]]}^{-1}\right|\right|_{\mathbf{L}(L^2(\overline{I}))}\leq 1\right\}, \quad (13)$$

where $||\overline{[f,[f,\not\Delta_I]]}^{-1}||_{\mathbf{L}(L^2(\overline{I}))}\leq 1$ *(which includes the requirement on the existence of* $\overline{[f,[f,\not\Delta_I]]}^{-1}$*) can be replaced by one of the following equivalent requirements*

$$[f,[f,\Delta_I]] \leq -I \quad (\textit{on } \mathcal{D}_{I,f,f}), \quad (14)$$

$$\overline{[f,[f,\Delta_I]]} \leq -I, \quad (15)$$

$$\sigma(\overline{[f,[f,\Delta_I]]}) \subset (-\infty,-1]. \quad (16)$$

Proof. First we show that under the assumption $[f,[f,\not\Delta_I]]\leq 0$ (which holds by (c) of Lemma 3.1 as $f\in\mathcal{C}_{[\mu_{\mathbf{g}}]}(I)$), the four requirements (14), (15), (16) and (R): "$\overline{[f,[f,\not\Delta_I]]}^{-1}$ exists and $||\overline{[f,[f,\not\Delta_I]]}^{-1}||_{\mathbf{L}(L^2(\overline{I}))}\leq 1$", are equivalent. The proof of the equivalence of (14), (15), (16) is

essentially the same used to prove the equivalence of the analogous three conditions in (c) of Lemma 3.1, we leave the details to the reader. Using the spectral representation of $\overline{[f,[f,\not\Delta_I]]}$, and viewing $\overline{[f,[f,\not\Delta_I]]}^{-1}$ as a spectral function of the former, (16) implies (R) straightforwardly. On the other hand, using the spectral theorem for $\overline{[f,[f,\not\Delta_I]]}^{-1}$, (R) implies that $\sigma(\overline{[f,[f,\not\Delta_I]]}) \subset (-\infty,-1] \cup [1,+\infty)$ (Use $\sigma(A) \subset [-||A||,||A||]$ and $\sigma(A^{-1}) \subset \overline{\{1/\lambda \mid \lambda \in \sigma(A) \setminus \{0\}\}}$ provided $0 \notin \sigma_p(A)$ this being our case when $A = \overline{[f,[f,\not\Delta_I]]}^{-1}$ because A admits inverse by construction). Then $[f,[f,\not\Delta_I]] \leq 0$, which is equivalent to $\sigma(\overline{[f,[f,\not\Delta_I t]]}) \subset (-\infty,0]$ by (c) of Lemma 3.1, entails $\sigma(\overline{[f,[f,\not\Delta_I]]}) \subset (-\infty,-1]$ which is (16). To conclude and prove (13) we reduce to the expression for $\mathbf{d}$ given in Theorem 2.2. The condition $\operatorname{ess\,inf}_{\overline{I}} |df| \geq 1$ which appears in the thesis of Theorem 2.2 is equivalent to $\operatorname{ess\,inf}_{\overline{I}} |df|^2 \geq 1$ which, in turn, is equivalent to

$$\operatorname{ess\,sup}\left\{|\mathbf{g}_x(\uparrow df,\uparrow df)|^{-1} \mid x \in \overline{I}\right\} \leq 1 \,. \tag{17}$$

Using the function $\mathbf{g}_x(\uparrow df,\uparrow df)^{-1} = -|\mathbf{g}_x(\uparrow df,\uparrow df)|^{-1}$ as a multiplicative (self-adjoint) operator on the whole space $L^2(\overline{I},\mu_{\mathbf{g}})$, (17) can equivalently be re-written

$$||\mathbf{g}_x(\uparrow df,\uparrow df)^{-1}||_{\mathbf{L}(L^2(\overline{I}))} \leq 1 \,. \tag{18}$$

On the other hand, holding $\mathbf{g}_x(\uparrow df,\uparrow df)^{-1} \cdot \mathbf{g}_x(\uparrow df,\uparrow df) = 1$ a.e., and $\mathbf{g}_x(\uparrow df,\uparrow df) = [f,[f,\not\Delta_I]]$ a.e., we also have

$$[f,[f,\not\Delta_I]] \circ \mathbf{g}_x(\uparrow df,\uparrow df)^{-1} = I_{L^2(\overline{I})} \,, \tag{19}$$

$$\mathbf{g}_x(\uparrow df,\uparrow df)^{-1} \circ [f,[f,\not\Delta_I]] = I_{\mathcal{D}_{I,f,f}} \,. \tag{20}$$

Notice that the closure of $[f,[f,\not\Delta_I]]$ is an operator because $[f,[f,\not\Delta_I]]$ is essentially self-adjoint ((b) of Lemma 3.1), moreover $\mathbf{g}_x(\uparrow df,\uparrow df)^{-1}$ is bounded by (18). These two facts together imply that $[f,[f,\not\Delta_I]]$ can be replaced by $\overline{[f,[f,\not\Delta_I]]}$ in both the identities above (also replacing $I_{\mathcal{D}_{I,f,f}}$ with the identity operator on the domain of $\overline{[f,[f,\not\Delta_I]]}$). Then, the uniqueness of the inverse operator implies that (18) is nothing but $||\overline{[f,[f,\not\Delta_I]]}^{-1}||_{\mathbf{L}(L^2(\overline{I}))} \leq 1$. $\Box$

Remark. Theorem 3.1 holds if replacing M with a vector fiber bundle $\mathfrak{F} \to M$ equipped with a positive Hermitean fiber-scalar product, and using a second-order differential operator working on compactly-supported almost-everywhere smooth sections, locally given by

$$\not\Delta^{(X,V)} = \frac{1}{2}\left[(\nabla^\mu - iX^\mu)(\nabla_\mu - iX_\mu) + V\right] \,.$$

X is any smooth Hermitean $SU(N)$-connection field V defining a Hermitean linear map $V_x : \mathcal{F}_x \to \mathcal{F}_x$ on each fiber $\mathcal{F}_x$, $x \in M$. This is because the identity (12) is preserved

$$[h,[f,\not\Delta^{(X,V)}]] = \mathbf{g}(\uparrow dh,\uparrow df)I \,, \tag{21}$$

where I is the fiber identity.

4 The algebraic point of view: generalizations towards a Lorentzian noncommutative geometry.

As found in Section 3, a generalization of the functional identity for the Riemannian distance exists in globally hyperbolic spacetimes. Here, we shall not attempt to give a complete investigation of noncommutative Lorentzian causal structures but we try to extract the algebraic content from the structure involved in the commutative case obtaining generalizations of the causal structure in both the commutative and noncommutative case. In particular we present a set of five axioms on *noncommutative causality* which give a straightforward generalization of the causal structure of globally hyperbolic spacetimes. We stress that there is no guarantee for the minimality of the presented set of axioms.

4.1.*Algebraic ingredients.* Assume that $(M, \mathcal{O}_t, \mathbf{g})$ is a globally hyperbolic spacetime and adopt all the notations and definition given in Section 1, 2 and 3 (including Appendix A). In particular we focus attention on the ingredients used to write down (4) from the point of view of C^*-algebra theory. A relevant mathematical object is the net of Hilbert subspaces, $\mathfrak{H} = \{\mathcal{H}_I \,|\, I \in \mathcal{X}\}$, where $\mathcal{H}_I = L^2(\overline{I}, \mu_{\mathbf{g}})$. $\mathfrak{H}$ enjoys several properties induced by the properties of the class of subsets $\mathcal{X}$ defined in Proposition 2.4. In the following $\leq$, used between elements of $\mathcal{X}$, indicates the partial ordering relation on $\mathcal{X}$ given by the set-inclusion relation. $(\mathcal{X}, \leq)$ is a direct set as shown in (a) of Proposition 2.4. We have a consequent trivial proposition concerning the elements of $\mathfrak{H}$.

Proposition 4.1. *Referring to the given definitions and notations,*
(a) *for any pair $I, J \in \mathcal{X}$ with $I \leq J$, $\mathcal{H}_I \subset \mathcal{H}_J$. More precisely, there is a Hilbert-space isomorphism from $\mathcal{H}_I$ onto a (closed) subspace of $\mathcal{H}_J$;*
(b) *$\mathfrak{H}$ is a direct set with respect that inclusion relation. More precisely, for any pair $I, J \in \mathcal{X}$ there is $K \in \mathcal{X}$ with $I, J \leq K$ such that $\overline{\mathcal{H}_I + \mathcal{H}_J} \subset \mathcal{H}_K$.*

A second set of relevant mathematical objects is given as follows. An elementary computation proves that if $f \in C(\overline{I})$, $C(\overline{I})$ denoting the commutative unital C^*-algebra of the continuous complex functions on $\overline{I}$, $||f||_\infty = ||O_f||_{\mathbf{L}(\mathcal{H}_I)}$, where O_f is the multiplicative operator $O_f h := f \cdot h$ for all $h \in \mathcal{H}_I$. Moreover the involution in $C(\overline{I})$, i.e. the complex conjugation $\overline{\cdot}$, is equivalent to the involution in $\mathbf{L}(\mathcal{H}_I)$, that is the Hermitean conjugation $\cdot^*$. Therefore $C(\overline{I})$ can be viewed as a subalgebra of the C^*-algebra of all bounded operators on $\mathcal{H}_I$, $\mathbf{L}(\mathcal{H}_I)$.
From now on we use the following notation $\mathfrak{A}_0 := \{\mathcal{A}_I\}_{I \in \mathcal{X}}$, where $\mathcal{A}_I$ denotes the commutative unital normed $*$-algebras containing all of multiplicative operators O_f, $f \in \mathcal{E}_{[\mu_{\mathbf{g}}]}(I)$. Moreover $\mathfrak{A} := \{\overline{\mathcal{A}_I}\}_{I \in \mathcal{X}}$ where $\overline{\mathcal{A}_I}$ indicates the C^*-algebra given by the Banach completion of $\mathcal{A}_I$.

Lemma 4.1 *Referring to the given definitions and notations, if $I \in \mathcal{X}$, $\overline{\mathcal{A}_I}$ is (isometrically) isomorphic to the C^*-algebra of the continuous functions on $\overline{I}$, $C(\overline{I})$.*

Proof. $C^\infty(\overline{I}) \subset \mathcal{A}_I$. $C^\infty(\overline{I})$ is $|| \cdot ||_\infty$-dense in $C(\overline{I})$ by Stone-Weierstrass' approximation theo-

rem because $C^\infty(\overline{I})$ and thus the closed sub $* - algebra$ of $C(\overline{I})$, $\overline{C^\infty(\overline{I})}$, separates the points of $\overline{I}$ and so $\overline{C^\infty(\overline{I})}$ must coincide with the algebra $C(\overline{I})$ it-self. $\Box$

Proposition 4.2. *Referring to the given definitions and notations, for $I, J \in \mathcal{X}$ and $I \leq J$, define $\Pi_{I,J}(a) := a\restriction_{\mathcal{H}_I}$, $a \in \overline{\mathcal{A}_J}$, then*
(a) $\Pi_{I,J}(\mathcal{A}_J) \subset \mathcal{A}_I$ *and thus $\Pi_{I,J}\restriction_{\mathcal{A}_J}: \mathcal{A}_J \to \mathcal{A}_I$ is a continuous (norm decreasing) unital $*$-algebra homomorphism;*
(b) $\overline{\Pi_{I,J}(\mathcal{A}_J)} = \overline{\mathcal{A}_I}$, *in other words $\Pi_{I,J} : \overline{\mathcal{A}_J} \to \overline{\mathcal{A}_I}$ is a surjective continuous unital C^*-algebra homomorphism.*

Proof. **(a)** can be proved by direct inspection using the fact that, in the sense of Lemma 4.1, $a\restriction_{\overline{I}} = a\restriction_{\mathcal{H}_I}$ where $a \in C(\overline{J})$ in the left-hand side is viewed as a function and $a \in \mathbf{L}(\mathcal{H}_J)$ in the right-hand side is viewed as a multiplicative operator. Let us prove **(b)**. $\overline{\Pi_{I,J}(\mathcal{A}_J)} = \overline{\mathcal{A}_I}$ and the surjectivity on $\overline{\mathcal{A}_I}$ of $\Pi_{I,J}$ to $\overline{\mathcal{A}_J}$ are trivially equivalent because $\Pi_{I,J}$ is continuous. We directly prove the surjectivity. Using Lemma 4.1, it is sufficient to show that, for every $f \in C(\overline{I})$ there is $g \in C(M)$ such that $g\restriction_{\overline{I}} = f$. Since M is Hausdorff, locally compact and $\overline{I}$ is compact, the existence of g follows from the Tietze extension theorem [24]. $\Box$

We have an immediate corollary:

Corollary. *In the hypotheses of Proposition 4.2, for $I, J, K \in \mathcal{X}$, $\Pi_{I,I} = Id$ and $I \leq J \leq K$ entails $\Pi_{I,K} = \Pi_{I,J} \circ \Pi_{J,K}$.*

The third ingredient is given by the class of causal functions. It takes the causal structure of the spacetime into account. Let us examine this ingredient from the algebraic point of view. First of all, notice that $I \in \mathcal{X}$ entails $\mathcal{C}_{[\mu_{\mathbf{g}}]}(\overline{I}) \subset \mathcal{A}_I$. From now on we use the notation $Co_I := \mathcal{C}_{[\mu_{\mathbf{g}}]}(\overline{I})$ and $\mathfrak{C} := \{Co_I\}_{I\in\mathcal{X}}$. Co_I is called the **causal cone** in $\mathcal{A}_I$.

Proposition 4.3. *Referring to the given definitions and notations, for $I, J \in \mathcal{X}$:*
(a) *Co_I is a convex cone containing the origin (i.e. $\alpha t + \beta t' \in Co_I$ for $\alpha, \beta \in [0, +\infty)$ and $t, t' \in Co_I$) whose elements are self-adjoint (i.e., $t \in Co_I$ implies $t^* = t$).*
(b) $[Co_I] := \{t_1 - t_2 + i(t_3 - t_4) \mid t_k \in Co_I, k = 1, 2, 3, 4\}$ *is a dense sub $*$-algebra of $\overline{\mathcal{A}_I}$;*
(c) $I \leq J$ *entails* $\Pi_{I,J}(Co_J) \subset Co_I$.

Proof. The only nontrivial statement is **(b)**, let us prove it. First notice that $[Co_I]$ is closed with respect to the algebra operations and $\mathbb{I} \in [Co_I]$, $\mathbb{I}$ being the unit of $\mathcal{A}_I$. Indeed, by the given definitions, $u, v \in [Co_I]$ entails $\alpha u + \beta v \in [Co_I]$ for all $\alpha, \beta \in \mathbb{C}$ and $u \in [Co_I]$ entails $u^* \in [Co_I]$. Then notice that $\mathbb{I}$ is nothing but the constant map $x \mapsto 1$ which is an element of $\mathcal{C}_{[\mu_{\mathbf{g}}]}(\overline{I}) = Co_I$. Moreover if $t \in Co_I$, since $\overline{I}$ is compact and t is continuous, there is $\alpha > 0$ such that if $t_\alpha := t + \alpha\mathbb{I}$, $t_\alpha(x) > 0$ for all $x \in \overline{I}$. So take $t, t' \in Co_I$ and define $t_\alpha, t'_{\alpha'} > 0$ as said. It is clear that $t_\alpha \cdot t'_{\alpha'} \in Co_I$ because the product of positive non-decreasing functions

is a non-decreasing increasing function. $t_\alpha \cdot t'_{\alpha'} \in Co_I$ means $t\cdot t' + \alpha\alpha'\mathbb{I} + \alpha t' + \alpha' t \in [Co_I]$, therefore the definition of $[Co_I]$ implies $t\cdot t' \in [Co_I]$. That result trivially generalizes to any pair $u, u' \in [Co_I]$. We have proven that $[Co_I]$ is a sub $*$-algebras of $\overline{\mathcal{A}_I}$. Now we prove that $[Co_I]$ separates the points of $\overline{I}$ and hence the closed algebra $\overline{[Co_I]}$ coincides with $\overline{\mathcal{A}_I} = C(\overline{I})$ because of Stone-Weierstrass' theorem. Let us show that, if $p, q \in \overline{I}$, there is $t_1 \in Co_I (\subset [Co_I])$ such that $t_1(p) \neq t_1(q)$. Indeed, if $p \prec q$, take $p' \lll p$, fix any $t \in Co_I$ and define $t_1 := t + \mathbf{d}(p', \cdot)\restriction_{\overline{I}}$. By (c) of Proposition 2.3, $t_1 \in Co_I$. Then $t_1(p) < t_1(q)$ by construction because $\mathbf{d}(p', p) < \mathbf{d}(p', q)$ by (b) of Proposition 2.3. If $p, q \in \overline{I}$ are spatially separated there is $p' \lll p$ with $q \notin J^+(p')$ (see the proof of Theorem 2.2). If $t \in Co_I$, take $\alpha \in [0, +\infty)$ with $t(p) + \alpha\mathbf{d}(p', p) > t(q)$. Then $t_1 := t + \alpha\mathbf{d}(p', \cdot)\restriction_{\overline{I}} \in Co_I$ and $t_1(p) \neq t_1(q)$. $\Box$

Remark. Since nontrivial causal functions cannot have compact support, we are forced to consider the unital normed $*$-algebras $\mathcal{A}_I$, as natural objects instead of the *nonunital* normed $*$-algebras $C_c(I)$ (the compactly-supported continuous functions on the open set I) if we want that some time function as $\mathbf{d}(p, \cdot)$ belongs to $\mathcal{A}_I$, as it results necessary from the proof of Proposition 4.3. On the physical ground this is related to the fact that a physical spacetime cannot be compact. A consequence of such a choice is that the class of C^*-algebras $\{\overline{\mathcal{A}_I}\}$ is not a *net* of C^*-algebras in the sense used in Quantum Field Theory [13] and it is not possible to define an overall C^*-algebra given by the inductive limit of the net.

The last ingredient we go to introduce is the class of densely-defined operators used in 3.2, $\mathfrak{G} := \{\mathbf{G}_I\}_{I\in\mathcal{X}}$, where $\mathbf{G}_I := \not{\Delta}_I : \mathcal{D}_I \to \mathcal{H}_I$ and $\mathcal{D}_I := C^\infty(\overline{I})$, $\mathcal{H}_I = L^2(\overline{I}, \mu_{\mathbf{g}})$. $\mathbf{G}_I$ will be said the **causal operators** on $\mathcal{H}_I$.

Proposition 4.4. *Referring to the given definitions and notations:*
(a) *for every $J \in \mathcal{X}$, $f, g \in Co_J$, there is a linear manifold $\mathcal{D}_{J,f,g} \subset \mathcal{D}_J$ such that:*
(i) $\mathcal{D}_{J,f,g}$ is dense in $\mathcal{H}_J$ and invariant with respect to $f, g, \mathbf{G}_I$;
(ii) if $K \in \mathcal{X}$, $K \leq J$, and $\Psi \in \mathcal{D}_{K,\Pi_{K,J}(f),\Pi_{K,J}(g)}$,

$$[f,\, [g\,,\, \mathbf{G}_J]]\,\Psi = [\Pi_{K,J}(f)\,,\, [\Pi_{K,J}(g)\,,\, \mathbf{G}_K]]\,\Psi\,; \tag{22}$$

(iii) $[f\,,\, [g\,,\, \mathbf{G}_J]]$ is essentially self-adjoint on $\mathcal{D}_{J,f,g}$;
(iv) if $\alpha, \beta > 0$, it holds

$$\mathcal{D}_{J,f,f} \cap \mathcal{D}_{J,g,g} \cap \mathcal{D}_{J,f,g} \cap \mathcal{D}_{J,g,f} \subset \mathcal{D}_{J,\alpha f+\beta g, \alpha f+\beta g}\,, \tag{23}$$

and $\mathcal{D}_{J,f,f} \cap \mathcal{D}_{J,g,g} \cap \mathcal{D}_{J,f,g} \cap \mathcal{D}_{J,g,f}$ is a core for $[\alpha f + \beta g,\, [\alpha f + \beta g,\, \mathbf{G}_J]]$;
(v) $[f\,,\, [g\,,\, \mathbf{G}_J]] \leq 0$ on $\mathcal{D}_{J,f,g}$.
(b) *$Co_{\mathbf{G}_K} := \{f \in Co_K \mid [f\,,\, [f\,,\, \mathbf{G}_K]] \leq -\gamma I$ for some $\gamma > 0\}$ is not empty.*

Proof. **(a)** (22) and (23) can be proven by direct inspection, $\mathcal{D}_{J,f,f} \cap \mathcal{D}_{J,g,g} \cap \mathcal{D}_{J,f,g} \cap \mathcal{D}_{J,g,f}$ is a core for $[\alpha f + \beta g,\, [\alpha f + \beta g, \mathbf{G}_J]]$ because that operator is essentially self-adjoint on that domain. This fact can straightforwardly be shown by following that way, based on Nelson's

theorem, used in the proof of Lemma 3.1. The remaining statements of the thesis are parts of the thesis of Lemma 3.1. **(b)** As the spacetime is globally hyperbolic there is a smooth time function t with dt everywhere timelike (see A.13). Therefore the smooth function $\mathbf{g}(\uparrow dt, \uparrow dt)$ is strictly negative on the compact $\overline{K}$ and thus posing $-\gamma := \max_{\overline{K}} \mathbf{g}(\uparrow dt, \uparrow dt)$, one has $\gamma > 0$ and $f := t\restriction_{\overline{K}} \in Co'_K$ because, by (12), $[f, [f, \mathbf{G}_K]] = \mathbf{g}(\uparrow dt, \uparrow dt) \leq -\gamma I$. □

Corollary. *With the hypotheses of Proposition 4.4, $t \in Co_{\mathbf{G}_K}$ entails $f + \alpha t \in Co_{\mathbf{G}_K}$ for every $f \in Co_K$ and $\alpha > 0$, in particular $Co_{\mathbf{G}_K}$ is a convex cone.*

Proof. Take $t \in Co_{\mathbf{G}_K}$. By (iv) in (a) of Proposition 4.4, if $A := \mathcal{D}_{K,t,t} \cap \mathcal{D}_{K,f,f} \cap \mathcal{D}_{K,f,t} \cap \mathcal{D}_{K,t,f}$

$$\overline{[f + \alpha t, [f + \alpha t, \mathbf{G}_K]]\restriction_A} = \overline{[f + \alpha t, [f + \alpha t, \mathbf{G}_K]]}\,.$$

In A, it also holds $[f+\alpha t, [f+\alpha t, \mathbf{G}_K]] = [f, [f\mathbf{G}_K]] + \alpha[f, [t, \mathbf{G}_K]] + \alpha[t, [f, \mathbf{G}_K]] + \alpha^2[t, [t, \mathbf{G}_K]]$. Finally (v) in (a) implies $[f + \alpha t, [f + \alpha t, \mathbf{G}_K]] \leq -\gamma I$ in the considered domain and so $\overline{[f + \alpha t, [f + \alpha t, \mathbf{G}_K]]} \leq -\gamma I$. In particular it holds in $\mathcal{D}_{K,f+\alpha t, f+\alpha t}$. □

Putting all together we can state the following **general algebraic hypotheses** which are fulfilled in the commutative case. However it is worthwhile stressing *they do not require the commutativity it-self explicitly* and thus they could be used in noncommutative generalizations. A fifth axiom will be introduced shortly.

(AH1) $\mathfrak{H} = \{\mathcal{H}_I\}_{I\in\mathcal{X}}$ *is a class of Hilbert spaces labeled in a partially-ordered direct set $(\mathcal{X}, \leq)$ such that (a), (b) of Proposition 4.1 is fulfilled.*
(AH2) $\mathfrak{A} = \{\mathcal{A}_I\}_{I\in\mathcal{X}}$ *is a class of unital sub $*$-algebras of the C^*-algebra of the bounded operators on $\mathcal{H}_I$, $\mathbf{L}(\mathcal{H}_I)$. $\overline{\mathcal{A}_I}$ denotes the unital C^*-algebra obtained as the Banach completion of $\mathcal{A}_I$ and we assume that (a), (b) of Proposition 4.2 holds (and thus its corollary holds too) with $\Pi_{I,J}$ defined therein.*
(AH3) $\mathfrak{C} = \{Co_I\}_{I\in\mathcal{X}}$*, with $Co_I \subset \mathcal{A}_I$, fulfills (a), (b) an (c) of Proposition 4.3.*
(AH4) $\mathfrak{G} = \{\mathbf{G}_I\}_{I\in\mathcal{X}}$*, with $\mathbf{G}_I : \mathcal{D}_I \to \mathcal{H}_I$ and $\mathcal{D}_I \subset \mathcal{H}_I$, is a class of densely-defined operators satisfying (a) and (b) of Proposition 4.4 (and thus its corollary).*

4.2. *Events, loci and causality.* Let us examine how the events of M and its topology arise in the algebraic picture introduced above. In particular, we show that the presented approach gives rise to a generalization of the concept of event in a spacetime, preserving the causal relations. When a manifold is compact, its points can be realized in terms of pure algebraic states on the C^*-algebra of continuous functions on the manifold [2, 20, 11]. If a manifold is *only* locally compact the construction is more complicated and involves irreducible $\mathbb{C}$-representations of the *nonunital* C^*-algebra of the functions which vanish at infinity[20, 6]. Here we want to develop an alternative procedure, involving pure states, which is useful from a metric point of view. We remind the reader that a linear functional $\omega : \mathcal{A} \to \mathbb{C}$ where $\mathcal{A}$ is a C^*-algebra, is said to be **positive** if $\omega(a^*a) \geq 0$ for all $a \in \mathcal{A}$. If $\mathcal{A}$ is unital, ω is said to be **normalized** if $\omega(\mathbb{I}) = 1$, $\mathbb{I}$

being the unit element of $\mathcal{A}$. In unital C^*-algebras, the positivity of a linear functional ω implies (a) the boundedness of ω and (b) $||\omega|| = \omega(\mathbb{I})$ (see proof of theorem 5.1 in [25]) so the normalization condition can be equivalently stated by requiring that $||\omega|| = 1$. A positive normalized linear functional on a unital C^*-algebra is called (**algebraic**) **state**. Concerning the GNS theorem we address the reader to [25] (theorem 5.1 in [25]) where a concise proof of that theorem is provided. In particular we remark that, by the GNS theorem, a positive normalized linear functional on a unital C^*-algebra is **real**, i.e., $\overline{\omega(a)} = \omega(a^*)$, and this implies that $\omega(a) \in \mathbb{R}$ if a is self-adjoint $a = a^*$. A state is said to be **pure** when it is an extremal element of the convex set of states. As is known, a state is pure iff it admits an irreducible GNS representation [25].

Proposition 4.5. *In our general algebraic hypotheses, let $\mathcal{S}_I$ denote the convex set of algebraic states λ_I on $\overline{\mathcal{A}_I}$, $I \in \mathcal{X}$ and let $\mathcal{S}_{pI} \subset \mathcal{S}_I$ denote the subset of pure states. Define the maps $J_{J,I} : \mathcal{S}_I \to \mathcal{S}_J$ with $J_{J,I} : \lambda_I \mapsto \lambda_I \circ \Pi_{I,J}$, where $I, J \in \mathcal{X}$ and $I \leq J$. Then,*
(a) *$J_{I,I} = Id$ and $J_{J,I}$ is injective if $I \leq J$.*
(b) *$J_{K,I} = J_{K,J} \circ J_{J,I}$ provided $I \leq J \leq K$.*

Proof. Everything is a trivial consequence of Proposition 4.2 and its corollary. □

As $\mathcal{X}$ is a direct set and Proposition 4.4 holds, it is natural to consider the *inductive limit* of the spaces $\mathcal{S}_I$ with respect to the maps $J_{I,J}$ and give the following definition. The definition of causal ordering $\trianglelefteq$ given below is a direct generalization of (c) in Proposition 2.4. After the introduction of generalization of the notion of event in terms of the notion of *locus* (also given in the definition below), it will be clear that $\trianglelefteq$ is nothing but a generalization of the causal partial ordering on the spacetime.

Definition 4.1. *In our general algebraic hypotheses and using the notation introduced above,*
(1) *a **locus** on $\mathfrak{A}$, Λ, is an element of the inductive limit of the class $\{\mathcal{S}_I\}_{I\in\mathcal{X}}$, with respect to the class of maps $\{J_{I,J}\}$. That is, Λ is an equivalence class of states in $\cup_{I\in\mathcal{X}}\mathcal{S}_I$ with respect to the equivalence relation*

$$\lambda_I \sim \lambda_J \quad \textit{iff there is } K \geq I, J \textit{ in } \mathcal{X} \textit{ with} \quad J_{K,I}(\lambda_I) = J_{K,J}(\lambda_J)\,; \tag{24}$$

$\mathfrak{L}$ denotes the space of loci , i.e., the inductive limit of the class $\{\mathcal{S}_I\}_{I\in\mathcal{X}}$.
(2) *$\Lambda \in \mathfrak{L}$ is said to be **pointwise** iff there is some pure state $\lambda_{I_0} \in \Lambda$. $\mathfrak{L}_p \subset \mathfrak{L}$ indicates the space of pointwise loci on $\mathfrak{A}$;*
(3) *$\Lambda \in \mathfrak{L}$ is said to **belong** to $I \in \mathcal{X}$, and we write $\Lambda\,\varepsilon\, I$, iff $\mathcal{S}_I \cap \Lambda \neq \emptyset$. In that case we define $\Lambda(f) := \lambda_I(f)$ for every $f \in \overline{\mathcal{A}_I}$ and $\lambda_I \in \Lambda \cap \mathcal{S}_I$;*
(4) *For $\Lambda, \Lambda' \in \mathfrak{L}$, we say that Λ' **causally follows** Λ, and we write $\Lambda \trianglelefteq \Lambda'$, iff $\Lambda(f) \leq \Lambda'(f)$ for every $I \in \mathcal{X}$ with $\Lambda, \Lambda'\,\varepsilon\, I$ and every $f \in Co_I$.*

Remark. Definition 4.1 is consistent, i.e. the equivalence relation preserves positivity and normalization. Indeed, for $I \leq K$, λ_I is respectively positive/normalized iff $J_{K,I}(\lambda_I)$ is respectively

such. We leave the trivial proof to the reader, based on the fact that $\Pi_{I,K}$ is a homomorphism of unital C^*-algebras. The well-definedness of $\Lambda(f)$ is proven in (c) below.

Proposition 4.6. *In our general algebraic hypotheses and using the notation introduced above, assuming $\Lambda, \Lambda', \Lambda'' \in \mathfrak{L}$ and $I, J, K \in \mathfrak{X}$, we have the following statements.*
(a) *If $\Lambda \,\varepsilon\, I$, there is only one $\lambda_I \in \Lambda \cap \mathfrak{L}_I$.*
(b) *$\Lambda \,\varepsilon\, I$ and $I \leq J$ entail $\Lambda \,\varepsilon\, J$. In that case $\lambda_J := J_{J,I}(\lambda_I) \in \Lambda \cap \mathcal{S}_J$ for $\lambda_I \in \Lambda \cap \mathcal{S}_I$ and $\Lambda(f) = \Lambda(\Pi_{I,J}(f))$ as $f \in \overline{\mathcal{A}_J}$.*
(c) *$\Lambda \in \mathfrak{L}_p$ iff every $\lambda_I \in \Lambda$ is a pure state. Hence $\mathfrak{L}_p$ is the inductive limit of the class $\{\mathcal{S}_{pI}\}_{I\in\mathfrak{X}}$ with respect to the class of the above-defined maps $\{J_{J,I}\}$, $I \leq J$ in $\mathfrak{X}$.*
(d) *$\Lambda \trianglelefteq \Lambda'$, $\Lambda' \trianglelefteq \Lambda''$ imply that $\Lambda(f) \leq \Lambda''(f)$ for every $f \in Co_I$ such that $\Lambda, \Lambda', \Lambda'' \,\varepsilon\, I$.*

Proof. **(a)** The thesis is a direct consequence of the injectivity of the maps $J_{I,K}$. **(b)** $\lambda_J \sim \lambda_I$ by construction. The remaining part is a direct consequence of the given definitions. Let us pass to prove **(c)**. If every $\lambda_I \in \Lambda$ is pure, $\Lambda \in \mathfrak{L}_p$ by definition, so consider the other case. Suppose there is a pure state $\lambda_I \in \Lambda$, we want to show that all the remaining states $\lambda_J \in \Lambda$ are pure too. By definition of locus there must be $K \in \mathfrak{X}$ with $I, J \leq K$ and $\lambda_J \circ \Pi_{J,K} = \lambda_I \circ \Pi_{I,K} =: \lambda_K$. GNS theorem (theorem 5.1 in [25]) and the surjectivity of $\Pi_{I,K}$ imply that if $\langle H, \pi, \Omega\rangle$ is a GNS triple for $\overline{\mathcal{A}_I}$ associated to λ_I, $\langle H, \pi \circ \Pi_{I,K}, \Omega\rangle$ is a GNS triple for $\overline{\mathcal{A}_K}$ associated to λ_K, and $\pi \circ \Pi_{I,K}$ is irreducible iff π is irreducible. Similarly if $\langle H', \pi', \Omega'\rangle$ is a GNS triple for $\overline{\mathcal{A}_J}$ associated to λ_J, $\langle H', \pi' \circ \Pi_{J,K}, \Omega'\rangle$ is another GNS triple for the same algebra $\overline{\mathcal{A}_K}$ associated to the same state λ_K and $\pi' \circ \Pi_{I,K}$ is irreducible iff π' is irreducible. Since (by GNS theorem) all GNS triples for an algebra ($\overline{\mathcal{A}_K}$) referred to a state (λ_K) are unitarily equivalent and the irreducibility is unitarily invariant, we conclude that π is irreducible iff π' is irreducible. This is the thesis. The proof of **(d)** is immediate by the given definitions and the item (b). $\Box$

The relationship between points on M and pointwise loci is established by the following theorem which does not require either the spacetime structure or a differentiable manifold structure. The only requirement is that M is a Hausdorff locally-compact topological space. More generally, the theorem shows that there is a bijection between loci on M and compactly-supported regular Borel probability measures μ with compact support on M. Such a bijective function reduces to a homeomorphism when restricted to the space of pointwise loci equipped with a suitable topology. We remind the reader that the support of a regular Borel measure is the complement of the largest open set with measure zero. Below $\int_M f\, d\mu$ is well defined by posing $f \equiv 0$ outside $\overline{J}$ since $\mathrm{supp}(\mu) \subset \overline{J}$.

Theorem 4.1. *Let M be a locally-compact Hausdorff topological space and $\mathfrak{X}$ a covering of M made of open relatively compact subsets and defining a direct set with respect to the set-inclusion relation. Define $\mathfrak{A} := \{\overline{\mathcal{A}_I}\}_{I\in\mathfrak{X}}$ with $\overline{\mathcal{A}_I} := C(\overline{I})$, $\mathcal{S}_I$, $\mathfrak{L}$ and $\mathfrak{L}_p$ as done in Def. 4.1, $\Pi_{I,J}(a) := a \restriction_{\overline{I}}$ and $J_{J,I}$ as in prop.4.3. Finally, denote the space of compact-support regular*

Borel probability measures on M by $\mathfrak{P}$. Consider the map $F : \mathfrak{P} \to \mathfrak{L}$, such that for $\mu \in \mathfrak{P}$,

$$F(\mu) := \left\{ \lambda_J^{(\mu)} \in \mathcal{S}_J \;\middle|\; J \in \mathcal{X} \text{ with } supp(\mu) \subset \overline{J},\ \lambda_J^{(\mu)}(f) := \int_M f\, d\mu \text{ for } f \in \overline{\mathcal{A}_J} \right\} .$$

(a) *F is well-defined, i.e., $F(\mu)$ is a locus for every $\mu \in \mathfrak{P}$. Moreover $F(\mu)\ \varepsilon\ I \in \mathcal{X}$ iff $supp(\mu) \subset \overline{I}$.*
(b) *F is bijective onto the set of the loci $\mathfrak{L}$.*
(c) *F restricted to the space of Dirac measures $\{\delta_x\}_{x\in M}$ gives rise to a homeomorphism from M onto $\mathfrak{L}_p$ equipped with the inductive-limit topology, every $\mathcal{S}_I$, $I \in \mathcal{X}$, being endowed with Gel'fand's topology.*

Proof. See the Appendix B. $\square$.

The following theorem proves that the relation $\trianglelefteq$ among loci is nothing but a generalization of the causal partial ordering on the spacetime.

Theorem 4.2. *In the hypotheses of theorem 4.1, also assume that $(M, \mathbf{g}, \mathcal{O}_t)$ is a globally hyperbolic spacetime and $\mathcal{X}$ is defined as in Proposition 2.4, $\mathfrak{C} = \{Co_I\}_{I\in\mathcal{X}}$ with $Co_I := T_{[\mathbf{g}]}(I)$. Consider the relation $\trianglelefteq$ defined in $\mathfrak{L}$ by Def. 4.1. and $\Lambda, \Lambda', \Lambda'' \in \mathfrak{L}$, then*
(a) *$\Lambda \trianglelefteq \Lambda'$ and $\Lambda' \trianglelefteq \Lambda$ together entail $\Lambda = \Lambda'$;*
(b) *if $\Lambda \trianglelefteq \Lambda'$ and $\Lambda' \trianglelefteq \Lambda''$ then, $\Lambda, \Lambda''\ \varepsilon\ I \in \mathcal{X}$ entails $\Lambda'\ \varepsilon\ I$ and thus $\trianglelefteq$ is transitive and defines a partial ordering relation on $\mathfrak{L}$;*
(c) *if F is that in Theorem 4.1, for every pair $x, y \in M$, $F(\delta_x) \trianglelefteq F(\delta_y)$ iff $x \preceq y$.*

Proof. See the Appendix B. $\square$.

Actually most of the content of Theorem 4.2 can be generalized using the general algebraic hypotheses as well as a further **causal convexity** axiom:

(AH5) *For $\Lambda, \Lambda', \Lambda'' \in \mathfrak{L}$, if $\Lambda \trianglelefteq \Lambda'$, $\Lambda' \trianglelefteq \Lambda''$ and $\Lambda, \Lambda''\ \varepsilon\ I \in \mathcal{X}$, then $\Lambda'\ \varepsilon\ I$.*

Notice that (AH5) is fulfilled in the globally-hyperbolic-spacetime case by (b) of Theorem 4.2.

Theorem 4.3. *In the general algebraic hypotheses, including the causal convexity axiom (AH5), and employing notations above, $\trianglelefteq$ is a partial-ordering relation in $\mathfrak{L}$.*

Proof. $\Lambda \trianglelefteq \Lambda$ is a trivial consequence of the definition of $\trianglelefteq$. The fact that $\Lambda \trianglelefteq \Lambda'$ and $\Lambda' \trianglelefteq \Lambda$ together entail $\Lambda = \Lambda'$ can be proven as in Theorem 4.2 where we have not used the spacetime structure. The transitivity of $\trianglelefteq$ follows from (AH5) and (d) of Proposition 4.6. $\square$

4.3. *Lorentzian distance.* We conclude by presenting a generalization of the Lorentzian distance in the general case. The following definition is very natural and can also be used in the gen-

eralized commutative case in the hypotheses of Theorem 4.2 concerning compactly supported probability measures on a spacetime. Notice that the definition makes sense by (AH3) and (AH4) which assure the existence of some function satisfying $[t,[t,\mathbf{G}_I]] \leq -I$ below.

Definition 4.2. *In the general algebraic hypotheses including the causal convexity axiom (AH5) and employing notations and conventions above, the* **Lorentzian distance** *of* $\Lambda, \Lambda' \in \mathfrak{L}$ *is*

$$\mathbf{D}(\Lambda, \Lambda') = \inf\left\{\langle \Lambda'(t) - \Lambda(t)\rangle \,\middle|\, t \in Co_I, \quad \Lambda, \Lambda' \,\varepsilon\, I \in \mathfrak{X}, \quad [t,[t,\mathbf{G}_I]] \leq -I \right\}, \tag{25}$$

where $\langle \alpha \rangle := \max\{0, \alpha\}$ *if* $\alpha \in \mathbb{R}$.

The item (iii) of (a) in (AH4) implies the following result, the proof being the same given for the corresponding part of Theorem 3.1.

Proposition 4.7. *In definition 4.2 the condition* $[t,[t,\mathbf{G}_I]] \leq -I$ *can be replaced by one of the three following conditions:*

$$\sigma(\overline{[t,[t,\mathbf{G}_I]]}) \subset (-\infty, -1]\,, \tag{26}$$

$$\overline{[t,[t,\mathbf{G}_I]]} \leq -I\,, \tag{27}$$

$$\overline{[t,[t,\mathbf{G}_I]]}^{-1} \quad \textit{exists and} \quad \left|\left|\overline{[t,[t,\mathbf{G}_I]]}^{-1}\right|\right|_{\mathbf{L}(\mathcal{H}_I)} \leq 1\,. \tag{28}$$

We have a conclusive theorem.

Theorem 4.4. *In the general algebraic hypotheses including the causal convexity axiom (AH5), employing notations and conventions above the Lorentzian distance enjoys the following properties for* $\Lambda, \Lambda', \Lambda'' \in \mathfrak{L}$.
(a) *In the hypotheses of Theorem 4.2 and assuming* $\mathbf{G}_I = \not{\Delta}_I$ *(defined in Theorem 3.1 for* $I \in \mathfrak{X}$*),* $\mathbf{D}(F(\delta_p), F(\delta_q)) = \mathbf{d}(p,q)$ *for every pair* $p, q \in M$.
(b) $0 \leq \mathbf{D}(\Lambda, \Lambda') < +\infty$. *In particular,* $\mathbf{D}(\Lambda, \Lambda') = 0$ *if either* $\Lambda = \Lambda'$ *or* $\Lambda \not\trianglelefteq \Lambda'$.
(c) *If* $\Lambda \trianglelefteq \Lambda' \trianglelefteq \Lambda''$ *then* $\mathbf{D}(\Lambda, \Lambda'') \geq \mathbf{D}(\Lambda, \Lambda') + \mathbf{D}(\Lambda', \Lambda'')$.

Proof. **(a)** The right-hand side of the definition of $\mathbf{D}(F(\delta_p), F(\delta_q))$ in (25) can be re-written as the right-hand side of (13) in Theorem 3.1. So the proof of the thesis is obvious. **(b)** The set in the right-hand side of (25) is not empty because, if $\Lambda \in \mathfrak{L}$, there is some $I \in \mathfrak{X}$ with $\Lambda \,\varepsilon\, \mathfrak{L}$ by definition of locus, moreover (AH4) implies that there is some $f \in Co_{\mathbf{G}_I} \neq \emptyset$ and thus $t = \alpha f \in Co_I$ and $[t,[t,\mathbf{G}_{I_t}]] \leq -I$ for some $\alpha > 0$. Then positivity and boundedness of $\mathbf{D}$ hold by definition. $\Lambda = \Lambda'$ implies $\mathbf{D}(\Lambda, \Lambda') = 0$ by the definition of $\mathbf{D}$. Finally suppose $\Lambda \not\trianglelefteq \Lambda'$. In that case there must exists $f \in Co_I$ for some $I \in \mathfrak{X}$ such that $\Lambda, \Lambda' \,\varepsilon\, I$ and $\Lambda(f) - \Lambda'(f) = \epsilon > 0$. Define $f_\nu := \nu f$, $f_\nu \in Co_I$ for all $\nu > 0$ because Co_I is a convex cone and $\Lambda(f_\nu) - \Lambda'(f_\nu) = \nu\epsilon$. Then Take $t_\gamma \in Co_{\mathbf{G}_I}$ (which exists by (AH4)) with $\gamma > 0$ such that $[t_\gamma,[t_\gamma,\mathbf{G}_I]] \leq -\gamma I$. Therefore, by (AH4) (and (iv), (v) of (a) in Proposition 4.4 and its corollary in particular), $t_\nu := f_\nu + (1/\sqrt{\gamma})t_\gamma$ is in Co_I as before and satisfies $[t_\nu,[t_\nu,\mathbf{G}_I]] \leq -I$. Finally

$\Lambda'(t_\nu) - \Lambda(t_\nu) = -\nu\epsilon + (1/\sqrt{\gamma})(\Lambda'(t_\gamma) - \Lambda(t_\gamma)) < 0$ if $\nu > 0$ is sufficiently large. Then the definition of $\mathbf{D}(\Lambda, \Lambda')$ gives $\mathbf{D}(\Lambda, \Lambda') = 0$. **(c)** Take $I \in \mathfrak{X}$ with $\Lambda, \Lambda'' \varepsilon I$. $\Lambda \trianglelefteq \Lambda' \trianglelefteq \Lambda''$ and (AH5) entail $\Lambda' \varepsilon I$ and thus $\Lambda''(f) - \Lambda(f) = (\Lambda''(f) - \Lambda'(f)) + (\Lambda'(f) - \Lambda(f))$ makes sense. In the given hypotheses, by (b), the identity can also be written $\langle \Lambda''(f) - \Lambda(f) \rangle = \langle \Lambda''(f) - \Lambda'(f) \rangle + \langle \Lambda'(f) - \Lambda(f) \rangle$. Using the definition of $\mathbf{D}$ it entails $\langle \Lambda''(f) - \Lambda(f) \rangle \geq \mathbf{D}(\Lambda, \Lambda') + \mathbf{D}(\Lambda', \Lambda'')$. Finally, since f is arbitrary, it implies the thesis. $\Box$

It is possible to define relations analogous to $\prec\!\!\prec$ and $\prec$ respectively, which we denote by $\lll\!\!\lhd$ and $\lhd$. $\Lambda \lll\!\!\lhd \Lambda'$ means $\mathbf{D}(\Lambda, \Lambda') > 0$, and $\Lambda \lhd \Lambda'$ means $\Lambda \trianglelefteq \Lambda'$ and $\Lambda \neq \Lambda'$ together. The final corollary shows that the content of A.7 can be restated in the general context without using causal path.

Corollary. *In the hypotheses of Theorem 4.4 and with the given definitions:*
(a) $\lll\!\!\lhd$ *and* $\lhd$ *are transitive and* $\Lambda \lll\!\!\lhd \Lambda'$ *implies* $\Lambda \lhd \Lambda'$ *;*
(b) *either* $\Lambda \lll\!\!\lhd \Lambda'$ *and* $\Lambda' \trianglelefteq \Lambda''$, *or* $\Lambda \trianglelefteq \Lambda'$ *and* $\Lambda' \lll\!\!\lhd \Lambda''$ *implies* $\Lambda \lll\!\!\lhd \Lambda''$.

Proof. **(a)** By (b) of Theorem 4.4, $\Lambda \lll\!\!\lhd \Lambda'$ entails $\Lambda \trianglelefteq \Lambda'$ and $\Lambda \lhd \Lambda'$, hence, by (c), $\lll\!\!\lhd$ is transitive. $\lhd$ is transitive too because of Theorem 4.3 and the definition of $\trianglelefteq$. **(b)** is a direct consequence of (c) in Theorem 4.4 and $\mathbf{D} \geq 0$. $\Box$

5 Open issues and outlook.

This paper shows that a generalization of part of the noncommutative Connes' program is possible in order to encompass Lorentzian and causal structures of (globally hyperbolic) spacetimes. However several relevant issues remain open. Obviously, first of all concrete models of the presented generalized formalism should be presented in the non-commutative case, moreover the minimality of the proposed axioms should be analyzed. An important point which should be investigated is the interplay between the topology of the space of loci and $\mathbf{D}$. In the commutative case and considering the events of a globally hyperbolic spacetime, $\mathbf{d}$ turns out to be continuous with respect to the topology of the manifold. Presumably a natural topology of the space of loci, in the general case, could be the inductive limit topology, each space $\mathcal{S}_I$ being equipped with the $*$-weak topology. One expects that $\mathbf{D}$ is continuous with respect such a topology. Another point is the following. We have focused attention on the Lorentzian generalization of (1) avoiding to tackle difficulties involved in possible generalizations of (3) which, presumably, should require a careful analysis of the spectral properties of the metric operators $\mathbf{G}_I$ introduced above. Such an analysis could reveals contact points with the content of [27] in spite of the evident differences of the presented approach and obtained results. Another important question which should be investigated concerns possible physical applications of the presented mathematical structure.

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Appendix A. Exponential map, Synge's world function, Spacetimes.

A.0. *Exponential map, Synge's world function.* Let $(M, \mathbf{g})$ be a smooth Riemannian or Lorentzian manifold. $\pi : TM \to M$ denotes the natural projection of TM onto M and, if $v \in TM$, $v_{\pi(v)}$ is the vector of $T_{\pi(v)}M$ associated to v. If $v \in TM$ and $\lambda \in \mathbb{R}$, λv is the element of TM with $\pi(\lambda v) = v$ and $(\lambda v)_{\pi(\lambda v)} = \lambda v_{\pi(v)}$.
Consider the map $(t, v) \mapsto \gamma(t, v) \in M$, where $\gamma(., v)$ is the unique geodesic which starts from $\pi(v)$ at $t = 0$ with initial tangent vector $v_{\pi(v)}$ and t belongs to the *maximal domain* (a_v, b_v) $(a_v < 0 < b_v)$. From known theorems on maximal solutions of (first order) differential equations on manifolds (TM) [21], the domain of γ, $\cup_{v\in TM}(a_v, b_v) \times \{v\}$ is open in $\mathbb{R} \times TM$ and γ is smooth therein. Then pick out the set $U \subset TM$ of elements v, such that $1 \in (0, b_v)$. It is possible to show that U is open. Notice that for each $v \in TM$, there is a sufficiently small $\lambda > 0$ such that $1 \in (0, b_{\lambda v})$, because of the identity $\gamma(\lambda t, v) = \gamma(t, \lambda v)$. From that identity one trivially proves that U is **starshaped**, i.e., if $v \in U$ then $\lambda v \in U$ for $\lambda \in [0, 1]$. The **exponential map**, $\exp : U \to M$, is defined as $\exp(v) := \gamma(1, v)$ [17]. Notice that $\exp \in C^\infty(U)$. If $p \in M$, $\exp_p$ denotes $\exp\restriction_{T_pM}$ and the open neighborhood of 0, $U_p := \{v \in U \mid \pi(v) = p\} \subset T_pM$, its natural domain. By direct inspection, one finds that $d\exp|_v \neq 0$ if v belongs to the zero section of TM. This entails that if one shrinks each U_p sufficiently about p to some starshaped and open neighborhood of p, $V_p \subset U_p$, $\exp_p\restriction_{V_p}$ defines a diffeomorphism from V_p to $\exp_p(V_p)$ which is open too. If $\{e_\alpha|_p\} \subset T_pM$ is a basis, $(t^1, \dots t^D) \mapsto exp_p(t^\alpha e_\alpha|_p)$, $t = t^\alpha e_\alpha|_p \in V_p$, defines a **normal coordinate system centred on** p. An open set $C \subset M$ is called a (**geodesically**) **convex normal** neighborhood if there is an open and *starshaped* set $W \subset TM$, with $\pi(W) = C$ such that $exp\restriction_W$ is a diffeomorphism onto $C \times C$. It is clear that C is connected and there is only one geodesic segment joining any pair $q, q' \in C$ which is *completely contained in* C, that is $t \mapsto exp_q(t((exp_q)^{-1}q'))$ $t \in [0, 1]$. It is possible to take C diffeomorphic to an open ball in $\mathbb{R}^{\dim M}$ [17]. Moreover if $q \in C$, $\{e_\alpha|_q\} \subset T_qM$ is a basis, $(t^1, \dots, t^D) \mapsto exp_q(t^\alpha e_\alpha|_q)$, $t = t^\alpha e_\alpha|_q \in W_q$ defines a *global* normal coordinate system onto C centred on q. The class of the convex normal neighborhood of a point $p \in M$ is not empty and defines a fundamental system of neighborhoods of p [17, 5, 21, 9].
In $(M, \mathbf{g})$ as above, $\sigma(x, y)$ indicates one half the squared geodesic distance of x from y, also known as **Synge's world function**: $\sigma(x, y) := \frac{1}{2}\mathbf{g}_x(exp_x^{-1}y, exp_x^{-1}y)$ [9]. By definition $\sigma(x, y) = \sigma(y, x)$ and σ turns out to be smoothly defined on $C \times C$ if C is a convex normal neighborhood. With the signature $(-, +, \cdots, +)$, we have $\sigma(x, y) > 0$ if the events are space-like separated, $\sigma(x, y) < 0$ if the events are time related and $\sigma(x, y) = 0$ if the events belong to a common null geodesic or $x = y$. All that and everything follows also holds in manifolds endowed with an Euclidean metric where σ (defined as above) is everywhere nonnegative. It turns out that [9] if γ is the unique geodesic from p to q in a convex normal neighborhood containing p, q,

with affine parameter $\lambda \in [0, l]$

$$\uparrow d_y\sigma(x,y)|_{y=\gamma(\lambda)} = \lambda\dot{\gamma}(\lambda)\,, \tag{29}$$

$$2\sigma(x,y) = \mathbf{g}_x(d_x\sigma(x,y), d_x\sigma(x,y)) = \mathbf{g}_y(d_y\sigma(x,y), d_y\sigma(x,y))\,. \tag{30}$$

A.1. *Lorentzian manifold.* A (smooth) **Lorentzian manifold** $(M, \mathbf{g})$ is a $n \geq 2$-dimensional smooth manifold M with a smooth Lorentzian metric $\mathbf{g}$ (with signature $(-, +, \cdots, +)$).

A.2. *Signature of vectors.* A vector $T \in T_xM$, $T \neq 0$, is said to be **space-like**, **time-like** or **null** if, respectively, $\mathbf{g}_x(T,T) > 0$, $\mathbf{g}_x(T,T) < 0$, $\mathbf{g}_x(T,T) = 0$. $T \neq 0$ is said to be **causal** if it is either time-like or null. The same nomenclature is used for co-vectors $\omega \in T^*_xM$ referring to $\uparrow\omega \in T_xM$, where $\mathbf{g}_x(\uparrow\omega, \cdot\,) = \omega$. If $T \in T_pM$, $|T| := \sqrt{|\mathbf{g}_p(T,T)|}$, similarly, if $\omega \in T^*_pM$, $|\omega| := \sqrt{|\mathbf{g}_p(\uparrow\omega, \uparrow\omega)|}$.

A.3. *Time orientation.* A Lorentzian manifold $(M, \mathbf{g})$ is said to be **time orientable** if it admits a smooth non vanishing vector field $Z \in TM$ which is everywhere time-like. A **time orientation**, $\mathcal{O}_t$, on a time-orientable Lorentz manifold, $(M, \mathbf{g})$, is one of the two equivalence classes of smooth time-like vector fields Z with respect to the equivalence relation $Z \sim Z'$ iff $\mathbf{g}(Z, Z') < 0$ everywhere. For each point $p \in M$, an orientations determines an analogous equivalence class of time-like vectors of T_pM, $\mathcal{O}_{tp}$. In a orientable Lorentz manifold, to assign a time orientation it is sufficient to single out a timelike vector in T_pM for a $p \in M$. With the given definitions, a causal vector (co-vector) $T \in T_pM$ ($\omega \in T^*_pM$) is said to be **future directed** if $\mathbf{g}_p(Z(p), X) < 0$ ($\mathbf{g}_p(Z(p), \uparrow\omega) < 0$). A causal vector (resp. covector) $T \in T_pM$ ($\omega \in T^*_pM$) is said to be **past directed** if $\mathbf{g}_p(Z(p), X) < 0$ ($\mathbf{g}_p(Z(p), \uparrow\omega) > 0$)

A.4. *Spacetime.* A **spacetime** $(M, \mathbf{g}, \mathcal{O}_t)$ is a Lorentzian manifold $(M, \mathbf{g})$ which is time orientable and equipped with a time orientation $\mathcal{O}_t$; the points of M are also called **events**.

A.5. *Regularity of curves and causal curves.* In a spacetime M, a **piecewise C^k curve** defined in a (open, closed, semi-closed) non-empty interval in $\mathbb{R}$, I, is a continuous map $\gamma : I \to M$ with a finite partition of I such that each subcurve obtained by restricting γ to each subinterval of the partition (*including its boundary*) is C^k. If the partition coincides with I it-self, the curve is said to be C^k. A piecewise C^1 curve γ is said to be **time-like**, **space-like**, **null**, **causal** if its tangent vector $\dot{\gamma}$ is respectively time-like, space-like, null, causal, *everywhere* in each subinterval I of the associated partition. A piecewise C^1 causal curve in a spacetime $\gamma : I \to M$ is said to be **future** (**past**) **directed** if if its tangent vector $\dot{\gamma}$ is **future** (**past**) **directed** *everywhere* in each subinterval I of the associated partition. In a spacetime M, if $p, q \in M$, a curve $\gamma : [a, b] \to M$ is said to be **from** p **to** q if $\gamma(a) = p$ and $\gamma(b) = q$.

A.6. *Continuous causal curves.* It is possible to extend the notion of causal future directed curves, considering **continuous future-directed causal curves** $\gamma : I \to M$. That is by requiring that, for each $t \in I$ there is a neighborhood of t, I_t and a convex normal neighborhood of $\gamma(t)$, U_t, such that, for $t' \in I_t \setminus \{t\}$, one has $\gamma(t') \neq \gamma(t)$ and there is a future-directed causal (smooth) geodesic segment $\gamma' \subset U_t$ from $\gamma(t)$ to $\gamma(t')$ if $t' > t$ there is a future-directed causal (smooth) geodesic segment $\gamma' \subset U_t$ from $\gamma(t')$ to $\gamma(t)$ if $t' < t$. Similar definitions hold concerning **continuous future-directed timelike curves**, by replacing "causal" with "timelike" in the definitions above. In this work a causal curve is supposed to be a continuous causal curve,

moreover continuous curves $\gamma : I \to M$ and $\gamma' : I' \to M$ are identified if there is an increasing homomorphism $h : I \to I'$ and $\gamma' \circ h = \gamma$.

A.7. *Causal relations of events.* In a spacetime $(M, \mathbf{g}, \mathcal{O}_t)$, if $p, q \in M$,
(i) $p \preceq q$ means that either $p = q$ or there is a future-directed causal curve from p to q,
(ii) $p \prec q$ means that $p \preceq q$ and $p \neq q$,
(iii) $p \lll q$ means that there is a future-directed time-like curve from p to q.
$\lll$ and $\preceq$ are clearly transitive.

Remark. In a spacetime $(M, \mathbf{g}, \mathcal{O}_t)$, if $p, q, r \in M$, $p \lll q$ and $q \preceq r$ entail $p \lll r$, and similarly $p \preceq q$ and $q \lll r$ entail $p \lll r$ [23].

A.8. *Causal sets.* We make use the following notations. Consider a spacetime $(M, \mathbf{g}, \mathcal{O}_t)$ and $S \subset M$, then

$$\begin{aligned} J^+(S) &:= \{q \in M \mid p \preceq q \text{ for some } p \in S\} \text{ the } \textbf{causal future} \text{ of } S\ , \\ J^+(S) &:= \{q \in M \mid q \preceq p \text{ for some } p \in S\} \text{ the } \textbf{causal past} \text{ of } S\ , \\ I^+(S) &:= \{q \in M \mid p \lll q \text{ for some } p \in S\} \text{ the } \textbf{chronological future} \text{ of } S\ , \\ I^-(S) &:= \{q \in M \mid q \lll p \text{ for some } p \in S\} \text{ the } \textbf{chronological past} \text{ of } S\ . \end{aligned}$$

Moreover $I(p, q) := I^+(p) \cap I^-(q)$ (which is not empty iff $p \lll q$) and $J(p, q) := J^+(p) \cap J^-(q)$. If $\emptyset \neq S \subset M$, $I^+(S)$ and $I^-(S)$ are open, $S \subset J^\pm(S) \subset \overline{I^\pm(S)}$, $I^\pm(S) = \mathrm{Int}(J^\pm(S))$ [21].

A.9. *Properties of $I^\pm(p)$ and $J^\pm(p)$.* (Theorem 8.1.2 in [28].) In a spacetime $(M, \mathbf{g}, \mathcal{O}_t)$, taking a sufficiently small normal convex neighborhood of $p \in M$, U, exp_p^{-1} defines a local diffeomorphism, $\phi : U \to \mathbb{R}^n$ with $\phi(p) = 0$, and $\phi(U \cap I^\pm(p)) = B \cap C$, where $B \subset \mathbb{R}^n$ is an open ball centred in 0 and C the open convex cone, with vertex 0, made of all the future directed timelike vectors. This result implies that both $I^\pm(p)$ and $J^\pm(p)$ are nonempty, connected by paths and connected.

A.10. *Causal relations of events again.* $p, q \in M$ are said to be **time related**, if either $I^+(p) \cap I^-(q) \neq \emptyset$ or $I^-(p) \cap I^+(q) \neq \emptyset$, **causally related** if either $J^+(p) \cap J^-(q) \neq \emptyset$ or $J^-(p) \cap J^+(q) \neq \emptyset$. Causally related events $p, q \in M$, $p \neq q$, which are not time related are called **null-related**. $S, S' \subset M$ are said to be **spatially separated** if $(J^+(S) \cup J^-(S)) \cap S' = \emptyset$ (which is equivalent to $(J^+(S') \cup J^-(S')) \cap S = \emptyset$).

A.11. *Causally convex sets, strongly causal spacetimes, Alexandrov topology.* In a spacetime M, we say that a set $S \subset M$ is **causally convex** when $J(p, q) \subset S$ if $p, q \in S$. It can be proven that an open set $U \subset M$ is causally convex iff for any future-directed causal curve γ and any choice of (continuous) parametrization $\gamma^{-1}(U)$ is open and connected in $\mathbb{R}$. The transitivity of $\preceq$ implies that $J^+(S)$, $J^-(S)$, $J(r, s)$ are causally convex for $\emptyset \neq S \subset M$ and $r \preceq s$. Also using the remark in A.7 one directly shows that $I^+(S)$, $I^-(S)$, $I(r, s)$ are causally convex for $\emptyset \neq S \subset M$ and $r \lll s$. A spacetime is **strongly causal** when every event admits a fundamental set of open neighborhoods consisting of **causally convex** sets. It is known that a spacetime M is strongly causal iff the **Alexandrov topology**, i.e., that generated by all the sets $I(p, q)$, $p, q \in M$, is the topology of M [23, 1].

A.12. *Globally hyperbolic spacetimes.* A **globally hyperbolic** spacetime (see the end of 8.3 in [28] about possible equivalent definitions) is a strongly-causal spacetime $(M, \mathbf{g}, \mathcal{O}_t)$ such that

every $J(p,q)$ is either empty or compact for each pair $p,q \in M$.
If the spacetime M is globally hyperbolic and $S \subset M$ is compact, $J^{\pm}(S) = \overline{I^{\pm}(S)}$ [28] and thus, using $I^{\pm}(S) = \mathrm{Int}(\overline{S})$ (A.8), $J^{\pm}(S) \setminus I^{\pm}(S) = \partial I^{\pm}(S) = \partial J^{\pm}(S)$. In particular $J^{\pm}(p) = \overline{I^{\pm}(p)}$.
A.13. *Stably causal spacetimes, global time functions.* A spacetime $(M, \mathbf{g}, \mathcal{O}_t)$ is said to be **stably causal** if there is a smooth map $f : M \to \mathbb{R}$ with df everywhere timelike (other equivalent definitions are possible [1]). A continuous map $t : M \to \mathbb{R}$ is said to be a **global time function** if strictly increases along every future-directed causal curve. A stably causal spacetime admits a global time function given by either $+f$ or $-f$, f being defined above. Remarkably, also the converse is true [12, 26]: If a spacetime admits a (global) time function, it admits a smooth map $f : M \to \mathbb{R}$ with df everywhere time-like.
A.14. *Causal spacetimes.* A spacetime is said to be **causal** if there are no events p,q such that $p \prec q \prec p$ (equivalently, it does not contain any closed causal curve). It is trivially proven that in a causal spacetime $\preceq$ is a *partial-ordering relation* in causal spacetimes. A spacetime is called **chronological** if there are no events p,q such that $p \prec\!\!\prec q \prec\!\!\prec p$ (equivalently, it does not contain any closed timelike curve).
A.15. *Implications of causal conditions.* It is known that [1, 21, 14]
globally hyperbolic $\Rightarrow$ **stably causal** $\Rightarrow$ **strongly causal** $\Rightarrow$ **causal** $\Rightarrow$ **chronological**.
In particular $\preceq$ is a partial-ordering relation in globally hyperbolic spacetimes too.
A.16. *Inextendible curves.* A causal curve $\gamma : I \to M$ is said to be future (past) **inextendible** if it admits no future (past) **endpoint**, i.e., $e \in M$ such that, for every neighborhood O of e, there is $t' \in I$ with $\gamma(t) \in O$ for $t > t'$ ($t < t'$). Any causal curve which admits an endpoint can be extended beyond that endpoint into a larger causal curve (only continuous in general). Hausdorff's maximality theorem implies that every (causal, timelike) curve can be extended up to a inextendible (causal, timelike) curve.
A.17. *Cauchy developments.* Let $S \subset M$ be any set in the spacetime $(M, \mathbf{g}, \mathcal{O}_t)$, $D^+(S)$ ($D^-(S)$) indicates the **future** (**past**) **Cauchy development of** S, i.e., the set of points p of the spacetime, such that every past (future) inextendible causal curve through p intersects S. (In particular $S \subset D^{\pm}(S)$.) $D(S) := D^+(S) \cup D^-(S)$ is the **Cauchy development of** S.
A.18. *Achronal and acausal sets.* A set $S \subset M$ is said to be **achronal** if $S \cap I^{\pm}(S) = \emptyset$ and **acausal** if $S \cap J^{\pm}(S) = \emptyset$. An achronal smooth spacelike embedded submanifold with dimension $\dim(M) - 1$ turns out to be also acausal ([21] p. 425).
A.19. *Cauchy surfaces.* A **Cauchy surface** (for M it-self), $S \subset M$, is a closed achronal set such that $D(S) = M$. There are different, also inequivalent definitions, of Cauchy surfaces, we use the definition of [28] which is equivalent to that given in [21] as stated in lemma 29 in chapter 14 therein.
A.20. *Globally hyperbolic spacetimes and Cauchy surfaces.* An important results states that: *a spacetime $(M, \mathbf{g}, \mathcal{O}_t)$ is globally hyperbolic iff it admits a Cauchy surface.* This statement can be adopted as an equivalent definition of a globally hyperbolic spacetime (see remark in the end of 8.3 in [28] for a proof of equivalence of the various definitions of globally hyperbolicity).
A.21. *Cauchy surfaces and global time functions in globally hyperbolic spacetimes.* All Cauchy surfaces of a globally hyperbolic spacetime M are connected and homeomorphic. M it-self is homeomorphic to $\mathbb{R} \times S$, S being a Cauchy surface of M and the projection map from M onto

$\mathbb{R}$ can be fixed to be a smooth global time function [23, 1, 28, 21].
A.22. *Smooth Cauchy surfaces.* The existence of spacelike smooth Cauchy surfaces in any globally hyperbolic spacetime is a very subtle issue. A a first, not complete, proof of existence of smooth Cauchy surfaces in general globally hyperbolic spacetimes is due to Dieckmann [3], however the complete proof, by Bernal and Sanchéz, is much more recent [4]. In (quantum) field theories, those are used to give initial data for hyperbolic field equations determining the dynamics of the fields everywhere in the spacetime [28, 29].
A.23. *Sets $I(S,p)$ and $J(S,p)$ and their properties.* In a globally hyperbolic spacetime $(M, \mathbf{g}, \mathcal{O}_t)$, if $S \subset M$ is a smooth Cauchy surface and $p \in J^+(S)$, $I(S,p)$ and $J(S,p)$ respectively denote $I^-(p) \cap I^+(S)$ and $J^-(p) \cap J^+(S)$. One can straightforwardly prove that $I(s,p)$ is not empty iff $p \in I^+(p)$. It is not so difficult to show that $I(S,p)$ and $J(S,p)$ are causally convex. A.8 implies that $I(S,p)$ is open and $I(S,p) \subset J(S,p)$. Finally, $J(S,p)$ is compact (theorem 8.3.12 in [28]) and $J(S,p) = \overline{I(S,p)}$ (Proposition 2.4). Analogous properties hold for the analogously defined regions $I(p,S)$ and $J(p,S)$.

Appendix B.

B.1. *Proof of Lemma 2.1.* In our hypotheses, $f \in \mathcal{C}_{[\mu_{\mathbf{g}}]}(\overline{I})$ is continuous on $\overline{I}$ and smooth in an open set $J := \overline{I} \setminus C = I \setminus C$ where $C \subset \overline{I}$ is closed with measure zero and $\partial I \subset C$. Notice that $\mu_{\mathbf{g}}(J) = \mu_{\mathbf{g}}(I) = \mu_{\mathbf{g}}(\overline{I})$ by construction. Therefore, concerning the first part of the thesis it is sufficient to show that, in $I \setminus C$, $\mathbf{g}(\uparrow df, \uparrow df) \leq 0$ and $\uparrow df$ is past direct if $\uparrow df \neq 0$. To this end, suppose $\mathbf{g}(\uparrow df, \uparrow df) > 0$ in $p \in I \setminus C$, then there must be an open neighborhood of p, U, where the same inequality holds. So define a smooth vector field T' which is timelike, future directed and orthogonal to $\uparrow df$ in U and $|T'| = 2|df|$, then define $T := T' - \uparrow df$. T is timelike and future-directed in U. If $\gamma : [0,1] \to U$ is a smooth integral curve of T, γ is timelike and future directed and it trivially holds $f(\gamma(1)) - f(\gamma(0)) = \int_0^1 \mathbf{g}_{\gamma(s)}(\uparrow df, \dot{\gamma}) ds < 0$. This is not allowed if f is a causal function in I. Similarly if $\uparrow df \neq 0$ is future directed at $p \in I \setminus C$, the same fact must hold in a neighborhood U of p. Take a local coordinate frame $x^1, \dots x^n$ in U where ∂_{x^1} is timelike, future directed and orthogonal to the spacelike vectors ∂_{x^k}, $k = 2, \dots n$. Obviously $\mathbf{g}(\partial_{x^1}, \uparrow df) < 0$. Let γ be an integral curve of ∂_{x^1} in U. γ is causal and future directed by construction and one gets the contradiction $f(\gamma(1)) - f(\gamma(0)) = \int_0^1 \mathbf{g}_{\gamma(s)}(\uparrow df, \dot{\gamma}) ds < 0$. Concerning (10) it is sufficient to prove it in I. Indeed, the thesis in $\overline{I} = I \cup \partial I$ is a direct consequence of the continuity of f and $\mathbf{d}$ in $\overline{I}$ and the fact that ∂I has measure zero. (In particular, if x or y or both belong to ∂I and $x \lll y$ there are two sequences $\{y_n\} \subset I$, $\{x_n\} \subset I$ with $x_n \to x$ and $y_n \to y$ as $n \to \infty$. The continuity of $\mathbf{d}$ implies that $x_n \lll y_m$ if n, m are sufficiently large and thus the right-hand side of (10) can be computed restricting to I.) Let us pass to prove that

$$\frac{f(y) - f(x)}{\mathbf{d}(x,y)} \geq \operatorname{ess\,inf}_I |df| \quad \text{for each pair } x, y \in I \text{ with } x \lll y\,. \tag{31}$$

To this end, fix $x, y \in I$ with $x \lll y$. Since the spacetime is globally hyperbolic there is a timelike future-directed segment geodesic $\gamma_0 : [0,1] \to M$ from x to y. This geodesic completely

belongs to I because I is causally convex. Using normal coordinates (see lemma 2.5 in Chapter 7 of [21]) about a geodesic segment γ_0', with $\gamma_0 \subset \gamma_0'$, it is possible to define a smooth variation of γ_0, $(t,s) \to \gamma_s(t)$ with $t \in [0,1]$, $\delta > 0$, $s \in D_1$, $\gamma_{s=0} = \gamma_0$, D_δ being the open disk in $\mathbb{R}^{\dim M-1}$ with radius $\delta > 0$ and centred in 0. It is possible to arrange $(t,s) \to \gamma_s(t)$ in order that (1) $(t,s) \to \gamma_s(t)$ with $(t,s) \in (0,1) \times D_1$ defines an admissible local coordinate map, (2) each curve γ_s is time-like and future-directed for $t \in (0,1)$ and admits t-limits towards 0^+ and 1^- defining smooth future directed causal curves from x to y. Notice that for every $s \in D_1$, $\gamma_s\restriction_{(0,1)} \subset I$ by construction. Take $s \in D_\delta$, $0 < \delta < 1$ and consider, for $t \in [0,1]$, $t \mapsto h_s(t) = f(\gamma_s(t))$. This function is non-decreasing and hence must admit derivative almost everywhere, such derivative is sommable and $f(y) - f(x) = h_s(1) - h_s(0) \geq \int_0^1 h_s'(\tau)d\tau$. The derivative is nonnegative and thus we may also write, using Fubini's theorem,

$$f(y) - f(x) \geq \frac{1}{\mathrm{vol}(D_\delta)} \int_{D_\delta} ds \int_0^1 d\tau h_s'(\tau) = \frac{1}{\mathrm{vol}(D_\delta)} \int_{[0,1]\times D_\delta} \frac{h_s'(\tau)}{\sqrt{|g(t,s)|}}\; d\mu_{\mathbf{g}}(t,s)\;.$$

$\mathrm{vol}(D_\delta)$ is the $R^{\dim M-1}$ volume of D_δ. In other words

$$f(y) - f(x) \geq \frac{1}{\mathrm{vol}(D_\delta)} \int_{[0,1]\times D_\delta} \frac{|\mathbf{g}_{\gamma_s(\tau)}(\uparrow df, \dot\gamma_s)|}{\sqrt{|g(t,s)|}}\; d\mu_{\mathbf{g}}(t,s)\;.$$

Barring vanishing measure sets, $\uparrow df$ is causal and past-directed or vanishes. Referring to an orthonormal basis of $T_{\gamma_s(t)}M$, $e_1, \ldots, e_D$, $(D = \dim M)$ where $e_1 = \dot\gamma_s(t)/|\dot\gamma_s(t)|$ is timelike, one straightforwardly proves that if $T \in T_{\gamma_s(t)}M$ is causal and future directed or vanishes then $|\mathbf{g}_{\gamma_s(t)}(T, \dot\gamma_s(t))| \geq |T||\dot\gamma_s(t)|$. Hence, posing $T = \uparrow d_{\gamma_s(t)} f$, we have

$$f(y) - f(x) \geq \frac{1}{\mathrm{vol}(D_\delta)} \int_{[0,1]\times D_\delta} \frac{|\mathbf{g}_{\gamma_s(\tau)}(\uparrow df, \dot\gamma_s)|}{\sqrt{|g(t,s)|}}\; d\mu_{\mathbf{g}}(t,s) \geq \frac{\mathrm{ess\,inf}_{[0,1]\times D_\delta}|df|}{\mathrm{vol}(D_\delta)} \int_{D_\delta} ds \int_0^1 |\dot\gamma_s(t)|\, dt$$

and thus,

$$f(y) - f(x) \geq \frac{(\mathrm{ess\,inf}_I|df|)}{\mathrm{vol}(D_\delta)} \int_{D_\delta} ds L(\gamma_s)\;.$$

Changing variables $s \to \delta\sigma$

$$f(y) - f(x) \geq \frac{(\mathrm{ess\,inf}_I|df|)}{\mathrm{vol}(D_1)} \int_{D_1} d\sigma L(\gamma_{\delta\sigma})\;.$$

Notice that $\sigma \mapsto L(\gamma_{\delta\sigma}) \leq L(\gamma_0) = \mathbf{d}(x,y)$ is continuous in D_1. Taking the limit as $\delta \to 0^+$ (using Lebesgue's dominate convergence theorem) we have

$$f(y) - f(x) \geq \frac{(\mathrm{ess\,inf}_I|df|)}{\mathrm{vol}(D_1)} \int_{D_1} d\sigma L(\gamma_0) = \frac{(\mathrm{ess\,inf}_I|df|)}{\mathrm{vol}(D_1)} \mathrm{vol}(D_1)\mathbf{d}(x,y)\;.$$

As $x \ll y$, $\mathbf{d}(x,y) > 0$ and thus

$$\inf\left\{ \frac{f(y) - f(x)}{\mathbf{d}(x,y)} \;\middle|\; x, y \in I,\;\; x \ll y \right\} \geq \mathrm{ess\,inf}_I|df|\;. \tag{32}$$

To conclude the proof it would be sufficient to show that, if f is a time function, for every $\epsilon > 0$ there are $x_\epsilon \lll y_\epsilon$ in I such that

$$\left| \frac{f(y_\epsilon) - f(x_\epsilon)}{\mathbf{d}(x_\epsilon, y_\epsilon)} - \operatorname{ess\,inf}_I |df| \right| < \epsilon \,. \tag{33}$$

To this end notice that, if $\operatorname{ess\,inf}_I |df| > 0$ there must be sequence $\{z_n\}_n \subset I$ such that each $\uparrow d_{z_n} f$ is time-like (and thus past-directed as we said above) and $|d_{z_n} f| \to \operatorname{ess\,inf}_I |df|$ as $n \to \infty$. In that case (33) is a consequence of the statement "for each z_n, and each $\mu > 0$ there are $x_{n,\mu} \lll y_{n,\mu}$ in I such that

$$\left| \frac{f(y_{n,\mu}) - f(x_{n,\mu})}{\mathbf{d}(x_{n,\mu}, y_{n,\mu})} - |d_{z_n} f| \right| < \mu \,. \tag{34}$$

Let us prove the statement above. Let $d_{z_n} f$ be timelike and past-directed. Define the normalized vector $e_1 = - \uparrow d_{z_n} f / |d_{z_n} f|$ and complete the basis of $T_{z_n} M$ with $D-1$ space-like vectors normalized and orthogonal to e_1. Finally consider the Riemannian normal coordinate system $\xi^1, \ldots, \xi^D$ centred on z_n generated by the basis $e_1, \ldots, e_D$. We restrict such a coordinate system in a sufficiently small convex normal neighborhood of z_n. By (d) of Proposition 2.1 if y has coordinates $\xi^1, \ldots, \xi^D$, $\xi^1 = \mathbf{d}(z_n, y)$. Then define $x_{n,\mu} = z_n$, and for every $y \equiv (t, 0, \ldots, 0)$ one has, by Lagrange's theorem (where $y' \equiv (t', 0, \ldots, 0)$ with $t' \in (0, t)$)

$$\frac{f(y) - f(x_{n,\mu})}{\mathbf{d}(x_{n,\mu}, y)} = \left| \frac{\partial f}{\partial \xi^1} \right|_{y'} \to \left| \frac{\partial f}{\partial \xi^1} \right|_{z_n} = |d_{z_n} f| \,,$$

as $t \to 0^+$, i.e., $y \to z_n$. For every $\mu > 0$, the existence of $x_{n,\mu} \lll y_{n,\mu}$ in I such that (34) is fulfilled follows trivially. The same proof can be used for the case $\operatorname{ess\,inf}_I |df| = 0$ provided that a sequence $\{z_n\}_n \subset I$ exists such that each $\uparrow d_{z_n} f$ is timelike and $|d_{z_n} f| \to 0$ as $n \to \infty$. Let us prove that such a sequence do exist if f is a time function. Suppose it is not the case and $\operatorname{ess\,inf}_I |df| = 0$. So it must happen that $|df| \equiv 0$ in some $E \subset I \setminus C$ with $\mu_{\mathbf{g}}(E) \neq 0$. In turn it implies that there is some $q \in I \setminus C$ where df_q is a null vector or vanishes. So, take a sequence of open neighborhoods of q, $U_i \subset I \setminus C$, where df is smoothly defined, such that $U_{i+1} \subset U_i$ and $\cap_i U_i = \{q\}$. If df_{q_i} is timelike for some $q_i \in U_i \setminus \{q\}$ for every i, the wanted sequence exists and this is assumed to be impossible by hypotheses. So it must be $|df| = 0$ in some U_{i_0}. But this is not possible too because, if $df_r \neq 0$ for some $r \in U_{i_0}$, the time-function f would be constant along a future-directed causal curve given by an integral curve of $\uparrow df$ in a neighborhood of r. Conversely, if $df \equiv 0$ in U_{i_0}, f would be constant along any timelike future directed curve in U_{i_0}. $\Box$

B.2. *Proof of Proposition 2.4.* It is convenient to prove **(d)** before because most of **(a)** is a straightforward consequence of the former. **(d)** If $S \subset M$ is a smooth Cauchy surface for M and $p \in I^+(S)$, the set $I(S, p)$ is open and causally convex (A.23). Since $\partial I^+(S) = S$, one has $\partial I(S, p) \subset S \cup \partial I^-(p)$ and hence $\partial I^+(S)$ has measure zero since S is a smooth hypersurface and (a) of Theorem 2.1 holds. To conclude that $I(S, p) \in \mathfrak{X}$, it is sufficient to show that

$\overline{I(S,p)} = J(S,p)$ noticing that the latter is compact and causally convex (A.23). Obviously, $\overline{I(S,p)} \subset J(S,p) = \overline{I^+(S)} \cap \overline{I^+(S)}$, so we have to show that $\overline{I^+(S)} \cap \overline{I^-(p)} \subset \overline{I(S,p)}$. Using the decomposition $\overline{I^+(S)} \cap \overline{I^-(p)} = (\partial I^+(S) \cap \partial I^-(p)) \cup (\partial I^+(S) \cap I^-(p)) \cup (I^+(S) \cap \partial I^-(p)) \cup (I^+(S) \cap I^-(p))$, one finds that the only thing to be shown is that $x \in \partial I^+(S) \cap \partial I^-(p)$ implies $x \in \overline{I^+(S) \cap I^-(p)}$. Take such an x. Notice that $x \in \partial I^+(S) = S$. Let B_x be an open neighborhood of x and γ a maximal causal geodesic segment from $x = \gamma(0)$ to $p = \gamma(1)$ which exists by (i) of Proposition 2.1. Extend γ into an inextendible causal geodesic γ'. Notice that γ (and γ') must be null, because $\mathbf{d}(x,p) = 0$ as a consequence of (a), (f) of Proposition 2.1 and A.12. Since $p \in I^+(S)$ and γ' intersects $S = \partial I^+(S)$ in x only (S being a Cauchy surface), $\gamma(t) \in I^+(S)$ for $t > 0$. It is possible to fix $t_0 > 0$, $t < 1$, such that $x' := \gamma(t_0) \in B_x$. Subsegments of a maximal geodesic segment are maximal and hence $x' \in \partial I^-(p)$. Therefore there is a sequence of points $\{x_n\} \subset I^-(p)$ with $x_n \to x'$ as $n \to \infty$. As $x' \in I^-(p), I^+(S), B_x$ and these sets are open, for some $N \in \mathbb{N}$ it must hold $x_n \in I^-(p), I^+(S), B_x$ if $n > N$. We have proven that for every open neighborhood B_x of x there is some $x_n \in I^-(p) \cap I^+(S) \cap B_x$. In other words, $x \in \overline{I^-(p) \cap I^+(S)}$. Let us pass to consider the open diamond regions $I(r,s)$. We want to show that, if $p \in M$, there is a fundamental set of open neighborhoods of p, $\{I(r_n, s_n)\} \subset \mathfrak{X}$ and in particular $\overline{I(r_n, s_n)} = J(r_n, s_n)$. From proposition 4.12 in [23] one finds that[5], in a globally hyperbolic spacetime M (it is sufficient the strongly causal condition), each point p admits a convex normal neighborhood U_p and an open neighborhood, A_p, such that (1) $A_p \subset U_p$ and (2) if $r, s \in A_p$, $I(r,s) \subset A_p$. In A_p, take a future directed geodesic segment through p, γ, and a two sequences of points on γ, $\{r_n\}$, $\{s_n\}$ such that $r_n \ll r_{n+1} \ll p \ll s_{n+1} \ll s_n$, and $r_n, s_n \to p$ as $n \to \infty$. As the spacetime is strongly causal (A.11), using the remark in A.7 one proves that $\{I(r_n, s_n)\}$ is a fundamental set of neighborhoods of p, and $J(r_{n+1}, s_{n+1}) \subset I(r_n, s_n) \subset U_p$. It is clear that each $I(r_n, s_n)$ is open, causally convex and $\partial I(r_n, s_n) \subset \partial I^+(r_n) \cup \partial I^-(s_n)$ has measure zero. As $J(r_n, s_n)$ is causally convex (A.11) and compact (A.12), to conclude it is sufficient to show that $J(r_n, s_n) = \overline{I(r_n, s_n)}$. Suppose this is not the case and thus there is $x \in J(r_n, s_n)$ with $x \notin \overline{I(r_n, s_n)}$. As is known, causal curves from r_n to s_n which are not smooth geodesic segments from r_n to s_n can be approximated by timelike curves from r_n to s_n [21]. This means that there must be a smooth null geodesic segment $\eta \subset J(r_n, s_n)$ from r_n to s_n with $x \in \eta$. Let us show that this is impossible. Indeed, since $r_n \ll s_n$, by (i) of Proposition 2.1 there is a timelike (and thus $\neq \eta$) geodesic segment η' from r_n to s_n and both η, η' must belong to the same geodesically convex neighborhood U_p.

(a) (ii) in (d) implies $\cup \mathfrak{X} = M$. Let us pass to show that $\mathfrak{X}$ is a direct set. We want to show that if $A, B \in \mathfrak{X}$, there is $C \in \mathfrak{X}$ with $A, B \subset D$. From now on $D := A \cup B$. As the spacetime is globally hyperbolic, it is homeomorphic to $\mathbb{R} \times S$, where $S \subset M$ is a smooth Cauchy surface (A.19-A.21). Then consider the natural smooth time function $t : M \to \mathbb{R}$ (which exists by A.13, A.15, A.21) associated to the former Cartesian factor. Take $t_1 < \min t\restriction_{\overline{D}}$ and $t_2 > \max t\restriction_{\overline{D}}$. The Cauchy surface $S_1 = \{x \in M \,|\, t(x) = t_1\}$ is in the past (with respect to t) of $\overline{D}$ and the Cauchy surface $S_2 := \{x \in M \,|\, t(x) = t_2\}$ is in the future of $\overline{D}$ and S_1. By definition of Cauchy surface, if

[5] The reader should pay attention to the fact that the definition of causally convex sets given in [23] is different from that used in this paper.

$p \in \overline{D}$, every inextendible future-directed timelike curve γ through p must intersect both S_1 and S_2. Let q the intersection of γ and S_2 and F the set of such points q. By construction, it holds $\overline{D} \subset \cup_{q \in F} I(S_1, q)$. Since $\overline{D}$ is compact we can extract a finite covering from that found above. In particular we have $\overline{D} \subset \cup_{i=1}^n I(S_1, q_i)$. To conclude define $C := \cup_{i=1}^n I(S_1, q_j)$. C is open because union of open sets, causally convex (if $p, q \in C$ satisfy $q \preceq p$, $p \in I(S_1, q_k)$ for some k and thus q and every causal curve from p to q belong to $I(S_1, q_k) \subset C$), $\partial C \subset \cup_{i=1}^n \partial I(S_1, q_j)$ has measure zero because every $I(S_1, q_j) \in \mathfrak{X}$ and thus $\partial I(S_1, q_j)$ has measure zero, $\overline{C} = \cup_{i=1}^n J(S_1, q_i)$ is compact (because union of compacts) and causally convex (the proof is similar to that for C). We conclude that $C \in \mathfrak{X}$ and $A, B \subset C$. **(b)** If f is a global smooth time function, which exists by A.13 and A.15 in globally hyperbolic spacetimes, and $A \in \mathfrak{X}$, then, trivially, $f\restriction_A \in \mathfrak{T}_{[\mu_{\mathbf{g}}]}(A)$ and $f\restriction_{\overline{A}} \in \mathfrak{C}_{[\mu_{\mathbf{g}}]}(\overline{A})$. **(c)** If $p, q \in \overline{I}$ and $p \preceq q$, it holds $f(p) \leq f(q)$ for all $f \in \mathfrak{C}_{[\mu_{\mathbf{g}}]}(I)$, $I \in \mathfrak{X}$ because $\overline{I}$ is causally convex. Let us prove that if $f(p) \leq f(q)$ for all essentially smooth causal functions f defined in any $\overline{I} \in \mathfrak{X}$ with $p, q \in \overline{I}$, then $p \preceq q$. Suppose that the implication is false. If p and q are spatially separated, as in the proof of Theorem 2.2 one finds two spatially separated, sufficiently small, regions $I(p', q')$ and $I(p'', q'')$ which respectively contain p and q and have spatially separated closures. By (c) of Proposition 2.3, $f_c : z \mapsto \mathbf{d}(p', z) + c\mathbf{d}(p'', z)$, $c > 0$ is an element of $\mathfrak{C}_{[\mu_{\mathbf{g}}]}(\overline{I})$ with $I = I(p', q') \cup I(p'', q'') \in \mathfrak{X}$. Moreover $f_c(q) = c\mathbf{d}(p'', q) < \mathbf{d}(p', p) = f_c(p)$ for c sufficiently small and this is a contradiction. If $q \prec p$, te map $f : z \mapsto d(x, z)$, defined on $J(x, y)$ with $x \prec\!\!\prec q$ and $p \prec\!\!\prec y$, produces a contradiction once again. $\Box$

B.3. *Proof of Theorem 4.1.* In the following we take advantage of Riesz' representation theorem [24] which proves that there is a bijective map $L \mapsto \mu_L$ between the set of positive linear functionals L on $C_c(\Omega)$, Ω being a locally compact topological space, and the set of regular Borel measures on Ω, such that $L(f) = \int_\Omega f \, d\mu_L$ for every $f \in C_c(\Omega)$.
(a) By (a) of Proposition 2.4, for every $\mu \in \mathfrak{P}$, there is $I \in \mathfrak{X}$ with $\text{supp}(\mu) \subset \overline{I}$. It is trivially proven that $\lambda_I^{(\mu)}$ is a state on $\overline{\mathcal{A}_I}$. Moreover, varying I, one obtains equivalent states since, by trivial properties of measures, $\text{supp}(\mu) \subset \overline{I}, \overline{J}$ entails $J_{K,I}(\lambda_I^{(\mu)}) = J_{K,J}(\lambda_J^{(\mu)})$ for $I, J \leq K$. We only have to show that if Λ_μ is the locus generated by some $\lambda_I^{(\mu)}$, every element $\lambda_J \in \Lambda_\mu$ must belong to $F(\mu)$. To this end assume that $\lambda_J \in \Lambda_\mu$, that is $\lambda_J \sim \lambda_I^{(\mu)}$. That equivalence relationship can be re-written as follows: for every $K \in \mathfrak{X}$ with $\overline{I} \cup \overline{J} \subset \overline{K}$ and for every $f \in C(\overline{K})$, it holds $\int_{\overline{K}} f d\mu = \lambda_J(f\restriction_{\overline{J}})$. Using the fact that $\text{supp}(\mu) \subset \overline{K}$ and (a) of Proposition 2.4, the obtained identity implies that if $h \in C_c(M)$ and $\text{supp}(h) \subset M \setminus \overline{J}$, $\int_M h \, d\mu = 0$. Uryshon's lemma [24] implies that $\mu(R) = 0$ for every compact $R \subset M \setminus \overline{J}$, then the regularity of μ implies that $\text{supp}(\mu) \subset \overline{J}$ and thus $\lambda_J \in F(\mu)$. We have proven that $F(\mu)$ is a locus, but also that $F(\mu) \, \varepsilon \, I$ implies $\text{supp}(\mu) \subset \overline{I}$. To conclude notice that if $\text{supp}(\mu) \subset \overline{I}$ then $F(\mu) \, \varepsilon \, I$ because $\lambda_I(\cdot) := \int_M \cdot \, d\mu \in F(\mu)$. **(b)** Injectivity: if $\mu \neq \mu'$, by Riesz' theorem there is $f \in C_c(M)$ with $\int_M f \, d\mu \neq \int_M f \, d\mu'$. Taking $K \in \mathfrak{X}$ with $\text{supp}(\mu), \text{supp}(\mu)' \subset K$ one gets $\lambda_K^{(\mu)}(f\restriction_{\overline{K}}) \neq \lambda_K^{(\mu')}(f\restriction_{\overline{K}})$ and thus $F(\mu) \neq F(\mu')$. Surjectivity: if Λ is a locus, take $I \in \mathfrak{X}$ with $\Lambda \, \varepsilon \, I$. Define $L_\Lambda(f) := \Lambda(f\restriction_{\overline{I}})$ for every $f \in C_c(M)$. L_Λ turns out to be a positive linear functional, therefore, Riesz' theorem proves that $L_\Lambda(f) = \int_M f \, d\mu_\Lambda$ for some regular

Borel probability measure μ_Λ with $\text{supp}(\mu_\Lambda) \subset \overline{I}$. Then consider the locus $F(\mu_\Lambda)$ and some $\lambda_J^{(\mu_\Lambda)} \in F(\mu_\Lambda)$. Tietze's extension theorem [24] entails $\lambda_J^{(\mu_\Lambda)} \in \Lambda$ and thus $F(\mu_\Lambda) = \Lambda$ by (a).
(c) First we prove that the elements of $F(\delta_x)$ are pure states. $f \in \overline{\mathcal{A}_I} = C(\overline{I})$ and $x \in \overline{I}$ imply that $\lambda_I^{(\delta_x)} : f \mapsto f(x)$ defines a pure state because $\langle \mathcal{H}, \pi, \Omega\rangle$ is a GNS triple for $\lambda_I^{(\delta_x)}$ if $\mathcal{H} = \mathbb{C}$, $\Omega = 1 \in \mathbb{C}$ and $\pi : f \mapsto f(x)$, and π is trivially irreducible. So $F(\delta_x)$ is a pointwise locus for every $x \in M$. F restricted to the space of Dirac measures $\{\delta_x\}_{x\in M}$ is surjective onto $\mathfrak{L}_p$. Indeed, take $\Lambda \in \mathfrak{L}_p$ and let $\lambda_I \in \Lambda \cap \mathcal{S}_{pI}$. As irreducible representations of a commutative C^*-algebras are unidimensional, a pure state ω on a commutative C^*-algebra admits a GNS representation on $\mathbb{C}$. As the cyclic vector is $1 \in \mathbb{C}$, one sees that ω is also multiplicative: $\omega(ab) = \omega(a)\omega(b)$. In other words ω it-self is an irreducible $\mathbb{C}$-representation of the C^*-algebra. Therefore λ_I is an irreducible representation of the commutative C^*-algebra $C(\overline{I})$. A known theorem in commutative C^*-algebras theory (e.g., see proposition 2.2.2 in [20]) implies that there is $x_\Lambda \in \overline{I}$ such that $\lambda_I(f) = f(x_\Lambda)$ for all $f \in C(\overline{I})$. (Precisely, the theorem states that $h_I : x \mapsto \lambda_I^{(\delta_x)}$ is a homeomorphism from $\overline{I}$ onto $\mathcal{S}_{pI}$ equipped with the Gel'fand topology.) Then consider $F(\delta_{x_\Lambda})$. It is clear that $\lambda_I \in F(\delta_{x_\Lambda})$ and thus $\Lambda = F(\delta_{x_\Lambda})$ by (a). Up to now we have proven that the map $H : M \to \mathfrak{L}_p$ with $H(x) = F(\delta_x)$ is a bijection from M onto $\mathfrak{L}_p$. It remains to show that H is a homeomorphism when $\mathfrak{L}_p$ is equipped by the inductive-limit topology obtained by equipping each $\mathcal{S}_I$ by the weak $*$-topology (Gel'fand topology). To prove that H is a homeomorphism, notice that M can be naturally identified to M', the inductive limit of the class of compact sets $\{\overline{I}\}_{I\in\mathcal{X}}$ equipped with a class of maps $F_{I,J} : \overline{J} \to \overline{I}$, when $J \subset I$, $F_{I,J}$ being the inclusion map. As $M \equiv M'$, the injective inclusion maps $F_I \mapsto M'$ ($F_I : x \mapsto [x]$ where $x \in \overline{I}$, $I \in \mathcal{X}$ and $[x] \in M' \equiv M$ being the equivalence class of x in the inductive limit) coincide with the usual inclusion maps of each $\overline{I}$ in M it-self. By definition the inductive-limit topology is the finest topology on the inductive limit set which makes continuous all the inclusion maps F_I. In other words a set $A \subset M' \equiv M$ is open iff $A \cap \overline{I}$ is open in the topology of $\overline{I}$, for all $I \in \mathcal{X}$. As the sets I are open and $\cup\mathcal{X} = M$, the inductive-limit topology on $M' \equiv M$ coincides to the original topology of M. To conclude, consider the following ingredients: the space $\mathfrak{L}_p$ realized as the inductive limit of the family $\{\mathcal{S}_{pI}\}_{I\in\mathcal{X}}$ (with maps $J_{I,J}$) equipped with the inductive-limit topology induced by the Gel'fand topology in the spaces $\mathcal{S}_{pI}$, the injective inclusion maps $G_I : \mathcal{S}_{pI} \to \mathfrak{L}_p$ and the homeomorphisms $h_I : \overline{I} \to \mathcal{S}_{pI}$ said above. Using (b) above, it is a trivial task to show that, for every $I \in \mathcal{X}$, $G_I \circ h_I = H \circ F_I$. As every $G_I \circ h_I$ is continuous, H turns out to be continuous. Conversely, since it also holds $H^{-1} \circ G_I = F_I \circ h_I^{-1}$ and every $F_I \circ h_I^{-1}$ is continuous, H^{-1} turns out to be continuous too. We have obtained that H is a homeomorphism. $\Box$

B.4. *Proof of Theorem 4.2.* **(a)** In the given hypotheses, take $I \in \mathcal{X}$ with $\Lambda, \Lambda' \varepsilon I$. If $\lambda_I \in \Lambda \cap \mathcal{S}_I$ and $\lambda'_I \in \Lambda' \cap \mathcal{S}_I$, one has $\lambda_I(t) = \lambda'_I(t)$, for every $t \in Co_I$. By (b) of Proposition 4.3, the linearity and the continuity of the states λ_I, λ'_I, one gets $\lambda_I = \lambda'_I$ and thus $\Lambda = \Lambda'$.
The proof of **(b)** and **(c)** is based on the following lemma.

Lemma B.1. *In the hypotheses of Theorem 4.2, $\Lambda \trianglelefteq \Lambda'$ implies both $supp(\mu') \subset J^+(supp(\mu))$ and $supp(\mu) \subset J^-(supp(\mu'))$, where $F(\mu) = \Lambda$ and $F(\mu') = \Lambda'$ and F is defined in Theorem 4.1.*

Proof of Lemma B.1. We prove $\text{supp}(\mu) \subset J^-(\text{supp}(\mu'))$, the other inclusion is analogous taking $p \lll q$ and $-\mathbf{d}(\cdot, q)$ in place of $\mathbf{d}(q, \cdot)$ in the following proof. Suppose that $\Lambda \trianglelefteq \Lambda'$ and there is $p \in \text{supp}(\mu)$ with $p \notin J^-(\text{supp}(\mu'))$. Since $J^-(\text{supp}(\mu'))$ is closed as $\text{supp}(\mu')$ is compact (see A.12), there is an open neighborhood of p which have no intersection with $J^-(\text{supp}(\mu'))$. Take $q \lll p$ in such a neighborhood. $J^+(q) \cap J^-(\text{supp}(\mu')) = \emptyset$ by construction. Let $I \in \mathfrak{X}$ be such that $\text{supp}(\mu), \text{supp}(\mu') \subset I$, such a set exists because of (a) of Proposition 2.4 and let $t \in Co_I$. By (c) of Proposition 2.3, $t_\alpha = t + \alpha \mathbf{d}(q, \cdot)) \restriction_{\overline{I}} \in Co_I$ for all $\alpha > 0$. Therefore $\Lambda \trianglelefteq \Lambda'$ entails, making use of Theorem 4.1, $\int_{\overline{I}} t_\alpha \, d\mu \leq \int_{\overline{I}} t_\alpha \, d\mu'$. Since $\alpha \mathbf{d}(q, \cdot)$ vanishes on $\text{supp}(\mu') \subset J^-(\text{supp}(\mu'))$, the same inequality can be re-written $\int_{\overline{I}} t \, d\mu + \alpha \int_{\overline{I}} \mathbf{d}(q, \cdot) \, d\mu \leq \int_{\overline{I}} t \, d\mu'$ for every $\alpha > 0$. On the other hand it holds $\int_{\overline{I}} \mathbf{d}(q, \cdot) \, d\mu > 0$. Indeed (1) if $U_p \subset \overline{U_p} \subset I^+(q) \cap I$ is an open neighborhood of p, it must be $\mu(U_p) \neq 0$ because $p \in \text{supp}(\mu)$, moreover (2) $\mathbf{d}(q, \cdot) \restriction_{\overline{U_p}} \geq \gamma > 0$ because of (a) of Proposition 2.1 and the continuity of $\mathbf{d}$. By consequence there must be some sufficiently large $\alpha > 0$ which produces a contradiction in $\int_{\overline{I}} t \, d\mu + \alpha \int_{\overline{I}} \mathbf{d}(q, \cdot) \, d\mu \leq \int_{\overline{I}} t \, d\mu'$. $\Box$

Let us come back to the main proof. Concerning **(c)**, the proof of the statement "$x \preceq y$ implies $F(\delta_x) \trianglelefteq F(\delta_y)$" is obvious by the given definitions. Using the proven lemma, the proof of the statement "$F(\delta_x) \trianglelefteq F(\delta_y)$ implies $x \preceq y$" is straightforward. Concerning **(b)**, notice that by the lemma and (a) of Theorem 4.1, $\Lambda \trianglelefteq \Lambda' \trianglelefteq \Lambda''$ implies, with obvious notations, $\text{supp}(\mu') \subset J^+(\text{supp}(\mu)) \cap J^-(\text{supp}(\mu''))$. Every open set $I \in \mathfrak{X}$ such that $\overline{I}$ contains both $\text{supp}(\mu)$ and $\text{supp}(\mu'')$ must contain $J^+(\text{supp}(\mu)) \cap J^-(\text{supp}(\mu''))$ because $\overline{I}$ is causally convex. Hence $\text{supp}(\mu') \subset J^+(\text{supp}(\mu)) \cap J^-(\text{supp}(\mu'')) \subset \overline{I}$. In other words, using (a) of Theorem 4.1, if $\Lambda \trianglelefteq \Lambda' \trianglelefteq \Lambda''$ and $\Lambda, \Lambda'' \, \varepsilon \, I \in \mathfrak{X}$, then $\Lambda' \, \varepsilon \, I$. $\Box$

Appendix C.

Proof of Theorem 2.1. The proof of Theorem 2.1 is based on two lemmata.

Lemma C.1. *If $(M, \mathbf{g}, \mathcal{O}_t)$ is globally hyperbolic and $p \in M$, referring to the definitions above, $C^+(p)$ and $\partial J^+(p) = \partial I^+(p) = J^+(p) \setminus I^+(p)$ are closed without internal points and have measure zero, finally $J^+(p) \setminus (C^+(p) \cup \partial J^+(p)) = I^+(p) \setminus C^+(p)$ is homeomorphic to $\mathbb{R}^{dim(M)}$.*

Proof. From now on $n := \dim(M)$ and $V_p^+ \subset T_pM$ is the cone made of future-directed causal vectors and 0. First consider $\partial J^+(p) = \partial I^+(p) = J^+(p) \setminus I^+(p)$, these identities being given in A.12. It is obvious that $\partial J^+(p)$ is closed, let us prove that it has measure zero. $J^+(p) \setminus I^+(p) \subset \exp_p(U_p \cap \partial V_p^+)$ where U_p is the open domain of the exponential map at p (see the Appendix A). Indeed if $q \in J^+(p) \setminus I^+(p)$, either $q = p$ or, by (i) of Proposition 2.1, there is a geodesic from p to q which must be null-like it being maximal and $q \notin I^+(p)$. Therefore (re-scaling the vector if necessary) there must be a vector $v \in \partial V_p^+ \cap U_p$ with $\exp_p v = q$. The Lebesgue measure of $\partial V_p^+ \subset \mathbb{R}^n$ vanishes and thus, since $\exp_p$ is smooth and thus locally Lipschitz, $\partial I^+(p)$ must have measure zero. Indeed one has that the part of $\partial I^+(p)$ contained in the domain V of any local coordinate chart (V, ψ) has measure zero, with respect to the

Lebesgue coordinate measure and thus $\mu_{\mathbf{g}}$, because $\psi \circ \exp_p$ is locally Lipschitz on $(\exp_p^{-1}(V))$ for all $k \in \mathbb{N}$. Then the countable measurability of $\mu_{\mathbf{g}}$ and the existence of a countable atlas of the manifold entails the thesis for the whole set $\partial I^+(p)$. The closure of $C^+(p)$ was proven in theorem 9.35 of [1], the absence of internal points is a trivial consequence of the measure zero (since nonempty open sets have positive measure $\mu_{\mathbf{g}}$). The last statement in the thesis is a consequence of proposition 9.36 in [1], due to Galloway. It remains to show that $C^+(p)$ has measure zero. Similarly to the proof for $\partial J^+(p)$, it is sufficient to prove that the Lebesgue measure in $\mathbb{R}^n$ of $\Gamma^+_{ns}(p)$ is zero: since $\Gamma^+_{ns}(p) \subset U_p$ and $C^+(p) = \exp_p(\Gamma^+_{ns}(p))$, the latter has measure zero if $\Gamma^+_{ns}(p)$ has measure zero. To this end notice that UM_p can be thought to be embedded in $\mathbb{R}^n$ and diffeomorphic to the intersection of the sphere S^{n-1}, $\sum_{i=1}^n (X^i)^2 = 1$ and the cone V^+, $X^1 \geq \sqrt{\sum_{i=2}^n (X^i)^2}$. UM_p is compact by construction. Fix $N \in \mathbb{N}$ and consider set $K_N := \{v \in UM_p \mid s_1(v) \leq N\}$. As s_1 is lower semicontinuous, K_N is closed and thus compact (since $K_N \subset UM_p$ which is compact and the topology being Hausdorff), moreover $\cup_N K_N = UM_p$. As a second step we define $S_N = \{v \in K_N \mid s_1(v)v \in U_p\}$. It is clear that, by the countability of the Lebesgue measure, the thesis is proven if one shows that, for every $N \in \mathbb{N}$, the image of the map $v \mapsto s_1(v)v$ with $v \in S_N$ has measure zero. The map $v \mapsto s_1(v)$, $v \in S_N$ is continuous (see the beginning of this appendix). Using the continuity of $v \mapsto s_1(v)v$ and the fact that U_p is open, it arises that S_N is open with respect to the topology of A_N. We conclude that $S_N = K_N \cap B_N$, where K_N is compact and B_N is an open set in S^{n-1}. If B_N is not connected we shall refer to each connected component of B_N in the following. The open set B_N admits a finite or countable class of components because the topology of S^{n-1} is second countable. Consider a countable class of compact sets $H_n \subset B_N$ such that $U_n H_n = B_N$ (they do exist because B_N is a connected manifold or it holds for each connected component). $H_n \cap K_N$ is compact (since the topology of S^{n-1} is Hausdorff), $K_N \cap H_n \subset S_N$ and $\cup_n (K_N \cap H_n) = S_N$. The function $v \mapsto s_1(v)v$ is continuous on each compact $K_N \cap H_n$ and thus its image has measure zero in $\mathbb{R}^n$. By countability the image of $v \mapsto s_1(v)v$, $v \in S_N$ has measure zero as required. $\Box$

The proof of the theorem ends by proving the lemma below. In the proof we make use of the sets

$$A_p := \{\lambda v \mid v \in UM_p,\ \lambda \in [0, s_1(v))\} \quad \text{and} \quad \mathcal{A}_p := A_p \setminus \{v \in T_pM \mid \mathbf{g}_p(v,v) = 0\}\,.$$

$A_p \subset U_p$ as one can straightforwardly prove using the definition of s_1 and the domain of the exponential map at p, U_p, moreover $\mathcal{A}_p$ is open as a consequence of the lower semicontinuity of s_1.

Lemma C.2. *If $(M, \mathbf{g}, \mathcal{O}_t)$ is globally hyperbolic and $p \in M$, referring to the definitions above the following statements hold true.*
(a) $\exp_p\restriction_{A_p}$ *is surjective onto* $J^+(p) \setminus C^+(p)$, $\exp_p\restriction_{\mathcal{A}_p}$ *is surjective onto* $I^+(p) \setminus C^+(p)$;
(b) $\exp_p\restriction_{A_p}$ *and* $\exp_p\restriction_{\mathcal{A}_p}$ *are smooth and injective;*
(c) $(\exp_p\restriction_{A_p})^{-1} \in C^\infty(J^+(p) \setminus C^+(p))$ *and* $(\exp_p\restriction_{\mathcal{A}_p})^{-1} \in C^\infty(I^+(p) \setminus C^+(p))$;
(d) *The map* $q \mapsto \mathbf{d}(p,q)^2$ *belongs to* $C^\infty(J^+(p) \setminus C^+(p))$;
(e) *The map* $q \mapsto \mathbf{d}(p,q)$ *belongs to* $C^\infty(I^+(p) \setminus C^+(p))$

(f) $\mathbf{g}_q(\uparrow d_q\mathbf{d}(p,q), \uparrow d_q\mathbf{d}(p,q)) = -1$ *for* $q \in I^+(p) \setminus C^+(p)$.

Proof. **(a)** Take $q \in J^+(p) \setminus C^+(p)$. If $q = p$, $q = \exp_p(0)$ and $0 \in A_p$. If $q \neq p$, by (i) of Proposition 2.1 (since the spacetime is globally hyperbolic) there is a future directed causal geodesic $\gamma : [0, b) \to M$ with $\gamma(0) = p$ and $\gamma(a) = q$, $a < b$ and γ is maximal from p to q. Rescaling the affine parameter of γ, we can assume that $v := \dot{\gamma}(0) \in UM_p$. It must hold $a \leq s_1(v)$ by maximality and $a \neq s_1(v)$ because it would imply $q \in C^+(p)$ by definition. Therefore $q = \exp_p(\lambda v)$ with $\lambda \in [0, s_1(v))$, namely, $q \in \exp_p(A_p)$. If $q \in J^+(p) \setminus C^+(p)$ but $q \notin \partial J^+(p)$ then (the spacetime being globally hyperbolic) $q \in I^+(p) \setminus C^+(p)$ and so v above must belong to $\mathcal{A}_p$. **(b)** As is known (see the Appendix A), the exponential map is smooth where it is defined. Let us consider the injectivity. Suppose there are $u, v \in A_p$ with $exp_p(u) = exp_p(v)$. This is equivalent to say that $exp_p(\lambda v_0) = exp_p(\mu u_0) = q$ for some $v_0, u_0 \in UM_p$ and $0 < \lambda < s_1(v_0)$, $0 < \mu < s_1(u_0)$. In other words q is contained in a maximal future-directed causal geodesic from p to some q' (after q), and thus the subsegment from p to q is a maximal geodesic, too. Moreover there is another maximal future-directed causal geodesic from p to q it-self. Lemmata 9.1 and 9.12 in [1] imply that q cannot be the image of a point in A_p and this is impossible. **(c)** It is a trivial consequence of (a), (b) and the fact that exp is a local diffeomorphism about every point of A_p. This is because there are no conjugate points with p along each future-directed causal geodesic starting from p before the corresponding cut point as stated in theorems 9.12 and 9.15 of [1]. **(d)** If $q \in J^+(p) \setminus C^+(p)$, there is a causal future-directed geodesic, γ, from p to q whose length coincides with $\mathbf{d}(p,q)$ and whose initial tangent vector is nothing but $\left(\exp\restriction_{A_p}\right)^{-1}(q) \in A_p$. Therefore $\mathbf{d}(p,q)^2 = L(\gamma)^2 = -\mathbf{g}_p\left(\left(\exp\restriction_{A_p}\right)^{-1}(q), \left(\exp\restriction_{A_p}\right)^{-1}(q)\right)$ from trivial properties of geodesics. From now on $-2\sigma(p,q)$ indicates the right-hand side of the obtained identity. **(e)** $\mathbf{d}(p,\cdot) = \sqrt{\mathbf{d}(p,\cdot)^2}$ and $x \mapsto \sqrt{x}$ is smooth for $x > 0$. $\mathbf{d}(p,\cdot)^2$ cannot vanish in the open set $I^+(p) \setminus C^+(p)$. **(f)** $d_q\mathbf{d}(p,q) = d_q\sqrt{-2\sigma(p,q)}$. Thus $d_q\mathbf{d}(p,q) = -(-2\sigma(p,q))^{-1/2}d_q\sigma(p,q)$ and by consequence one gets $\mathbf{g}_q(\uparrow d_q\mathbf{d}(p,q), \uparrow d_q\mathbf{d}(p,q)) = (-2\sigma)^{-1}\mathbf{g}_q(\uparrow d_q\sigma(p,q), \uparrow d_q\sigma(p,q))$. (30) holds in geodesically convex neighborhoods. However it can also be proven in our hypotheses following the proof of theorem 1.2.3, items (iii) and (iv), in [9] which only employs the variational definition of (timelike) geodesics. Using (30) one has $\mathbf{g}_q(\uparrow d_q\mathbf{d}(p,q), \uparrow d_q\mathbf{d}(p,q)) = -1$. $\Box$ $\Box$

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